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16. Abstract

This report presents a new boundary condition estimation framework for transportation networks in which the state is modeled by a first order scalar conservation law. Using an equivalent formulation based on a Hamilton-Jacobi equation, we pose the problem of estimating the boundary conditions of the system on a network, as a Mixed Integer Linear Program (MILP). We show that this framework can handle various types of traffic flow measurements, including floating car data or flow measurements. To regularize the solutions, we propose a compressed sensing approach in which the objective is to minimize the variations over time (in the L_1 norm sense) of the boundary flows of the network. We show that this additional requirement can be integrated in the original MILP formulation, and can be solved efficiently for small to medium scale problems.

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Boundary Conditions Estimation on a Road Network Using Compressed Sensing

by

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Report SWUTC/16/600451-00090-1
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Disclaimer

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Contents

Executive summary

Boundary flows are a critical piece of data required in predicting the evolution of traffic flow over some time horizon. They are sometimes encoded as Origin-Destination (O/D) matrices, and represent the future loading of the network. This piece of knowledge is very important for both estimation of traffic (in particular forecast), routing, and traffic flow control, for example model predictive control involving the optimization of traffic over a finite horizon.

This report presents a new boundary condition estimation framework applicable to transportation networks in which the state is modeled by a first order scalar conservation law. This framework is based on the classical Lighthill Whitham Richards (LWR) traffic flow model, which is here equivalently described as a Hamilton-Jacobi equation. Inspired by previous work on traffic flow control over networks, we formulate the problem of estimating the boundary conditions of the system over a network, as a Mixed Integer Linear Program (MILP), or Linear Program (LP). We show that this framework can handle various types of traffic flow measurements, including floating car data or flow measurements. To regularize the solutions, we propose a compressed sensing approach in which the objective is to minimize the variations over time (in the L_1 norm sense) of the boundary flows of the network. We show that this additional requirement can be integrated in the original MILP formulation, and can be solved efficiently for small scale problems.

We solved the boundary flow estimation problem on two example problems involving simulated initial density data, though floating car data can be incorporated in this framework as internal boundary conditions. With the compressed sensing term regularizing the flow estimates, this algorithm gives the simplest solution (in the sense of solution with the least amount of features) to the problem that satisfies both the constraints of the model and the data.

Significant challenges remain for scalability to practical networks, which can involve tens to hundreds of thousands of links, and which are not tractable with the current formulation. Future work will focus on these issues, and on the possible integration of non-model based approaches with the algorithm investigated in the present report.

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0.1 Introduction

Physical systems driven by *Partial Differential Equations* (PDEs) are very common in engineering, biology, physics and chemistry. Such systems are often referred to as *Distributed Parameter Systems*, as their state is infinite dimensional. In the context of transportation engineering, the flow of traffic is usually modeled as such, using various classes of PDEs, most notably first [24, 28] and second order [2, 5, 15, 14] traffic flow models.

Among these distrubted parameter systems (which can be modeled by the previously cited PDEs), the transportation network is of critical importance, and its performance has a dramatic impact on both our lives and the economy. Traffic congestion is a considerable issue worldwide and is expected to become worse as global traffic demand increases. With a yearly cost estimated at $700 per driver in the USA, congestion costs more than one week of wages for the average American. It also has a significant impact on air pollution, and causes an estimated 2% increase in fuel consumption, in addition to the economic losses induced by delays.

Reducing traffic congestion can be achieved by increasing the capacity of roads (which is expensive), modifying user demand, or actively controlling traffic using for instance adaptive speed limits, ramp metering, or traffic signal control. Traffic flow control [8, 27] is a very promising direction for reducing traffic congestion, as it does not require expensive road construction, and does not require the control of user demand, which is sometimes impractical (for instance during work hours). However, traffic control also requires to precisely estimate the boundary flows that will apply to the network, a problem known as *Origin-Destination* (OD) matrix estimation.

While a large body of literature focuses on the problem of OD matrix estimation [31, 29, 30, 19], little has been done in fusing real-time measurements into a model-based framework to estimate the possible boundary conditions. Indeed, one of the difficulty of the Origin-destination estimation problem is the sheer number of variables, and the low amount of available data, which makes it a greatly underdetermined problem. Integrating models with large-scale systems is also complex, since the computational time required to process the large scale datasets associated with O/D estimation problems can make the problem intractable.

In this report, we propose to address both issues outlined above by developing a framework based on the Lighthill Whitham Richards (LWR) [24, 28] Partial Differential Equation (PDE) with triangular flux function on transportation networks. Using an equivalent *Hamilton Jacobi* formulation of the LWR PDE, we show that the problem of estimating the boundary flows at the boundaries of a link can be posed as a *Mixed Integer Linear Program* (MILP). We then extend this formulation to the network problem (with junctions), and show that the global boundary flow estimation problem is also a LP or a MILP, and can be solved efficiently on a regular desktop computer, yielding an optimal solution to the estimation problem. Given that the problem to solve is greatly underdetermined, we propose a compressed sensing [20] approach to regularize the solutions, by minimizing the L_1 norm of the flow variations across the boundaries.

The rest of this report is organized as follows. Section 0.2 reviews the framework for the traffic estimation developed in [6] using the LWR traffic flow model. We then derive the optimal boundary flow estimation problem as a LP (or a MILP depending on the objective function used in the problem) using the traffic density estimation investigated earlier. Section 0.5 presents the boundary flow estimation problem on a single highway link. We then generalize this problem in Section 0.6 to the boundary flow estimation of a general highway network containing multiple mainstreams, junctions and on/off-ramps. Flow conservation constraints at the junctions are defined using a set of equalities derived from [22], and are incorporated in the control problem, resulting again in a LP or a MILP.

0.2 Model Definition

0.2.1 Traffic flow model

In this report, we define one link on a highway section as $P := [\xi, \chi]$ where ξ and χ respectively represent the upstream and downstream post miles of the link on the highway. This link can be described by a density function $\rho(t, x)$, which depends on both space and time, and which evolves according to a traffic model. One of the most commonly used macroscopic models for the traffic flow is the *Lighthill-Whitham-Richards* (LWR) model [24], [28]. Since the state itself is a function, this model is a distributed parameter systems, encoded by the following Partial Differential Equation (PDE):

$$\frac{\partial \rho(t,x)}{\partial t} + \frac{\partial \psi(\rho(t,x))}{\partial x} = 0 \tag{1}$$

While the description of the state by the density function is widely adopted, the LWR PDE exhibit shocks (discontinuities), which are features inherent to hyperbolic PDEs. To avoid dealing with these discontinuities, we can alternatively describe traffic as a scalar function $\mathbf{M}(\cdot,\cdot)$, known as the *Moskowitz function* [25], [26], which is obtained by integration of the density function. The Moskowitz function satisfies the following *Hamilton-Jacobi*(HJ) PDE, obtained by integration of the LWR PDE:

$$\frac{\partial \mathbf{M}(t,x)}{\partial t} - \psi\left(-\frac{\partial \mathbf{M}(t,x)}{\partial x}\right) \;=\; 0 \tag{2}$$

The Moskowitz function is also known as the Cumulative Number of Vehicles function, and can be thought of as follows. Let us label vehicles on the stretch of highway $[\xi, \chi]$ with an increasing order, with respect to a given reference vehicle. The Moskowitz function $M(t, x)$ represents (up to a constant of integration) the label of the vehicle that is closest to location x, at time t. While this function is technically a discontinuous function, solutions to equation (2) are typically continuous, which corresponds to a continuum approximation of traffic. In the above equation, the function $\psi(\cdot)$ is known as the *Hamiltonian*, of *Fundamental Diagram.* In the remainder of this report, the *Hamiltonian* is assumed to be the following continuous and concave triangular function:

$$\psi(\rho) = \begin{cases} v_f \rho & : \rho \in [0, \rho_c] \\ w(\rho - \kappa) & : \rho \in [\rho_c, \rho_\kappa] \end{cases} \tag{3}$$

where ρ_c and ρ_κ are respectively the critical and maximal traffic density related by the following formula.

$$k_c = \frac{-w\rho_\kappa}{v_f - w}$$

The capacity of the stretch of highway (which corresponds to the maximal throughput) is $\rho_c \cdot v_f$. While the method investigated in this report could be applied to general concave flux functions, a piecewise linear flux function allows us to pose the control problem as a *Mixed Integer Linear Program* (MILP), while general concave flux functions would result in a *Mixed Integer Convex Program*, for which no efficient computational methods are currently available. Note that triangular fundamental diagrams are very commonly used in traffic [18], since they are very robust and depend on a minimum number of parameters (3) that are easy to physically interpret.

0.2.2 Solutions to the Lighthill-Whitham-Richards equation

One of the several classes of weak solutions to equation (2) is known as the *Barron-Jensen/Frankowska* (B-J/F) solution [3], [21], which are investigated in this report. The B-J/F solutions to equation (2) are fully characterized by the *Lax-Hopf* formula , which was initially derived using the control framework of viability theory [4]. Note that B-J/F solutions to Hamilton Jacobi PDEs are equivalent to viscosity solutions [17], which are continuous, if no internal conditions are considered as part of the solution. If internal conditions are considered in the problem, the B-J/F solutions to the PDE are lower-semicontinous in general.

Solving partial differential equations requires the definition of initial, boundary and internal conditions, which can be encoded as the general concept of value conditions [11]. Such conditions encode a constraint on the value of the solution to the HJ PDE (2), these constraints being set on a line segment of the space-time domain.

Definition 1 (Value Condition) *A value condition* $\mathbf{c}(\cdot,\cdot)$ *is a lower semi-continuous function defined on a subset of* $[0, t_{max}] \times [\xi, \chi]$.

In this report, we assume the value conditions are affine functions of space and time, defined on a line segment of $\mathbb{R}_+ \times X$. Given an arbitrary value condition $\mathbf{c}(\cdot,\cdot)$, we define its associated solution $\mathbf{M_c}(\cdot,\cdot)$ to HJ PDE (2) using the following Lax-Hopf formula [1].

Proposition 1 (Lax-Hopf formula) *Let* $\psi(\cdot)$ *be a concave and continuous Hamiltonian, and let* $\mathbf{c}(\cdot,\cdot)$ *be a value condition, as in Definition 1. The B-J/F solution* $\mathbf{M_c}(\cdot,\cdot)$ *to (2) associated with* $\mathbf{c}(\cdot,\cdot)$ *is defined algebraically by:*

$$\mathbf{M_c}(t,x) = \inf_{(u,T)\in \mathrm{Dom}(\varphi^*)\times\mathbb{R}_+} \left(\mathbf{c}(t-T, x+Tu) + T\varphi^*(u)\right) \tag{4}$$

where the function $\varphi^*(\cdot)$ *is the Legendre-Fenchel transform of the upper semicontinuous Hamiltonian* $\psi(\cdot)$*, given by:*

$$\varphi^*(u) := \sup_{p\in \mathrm{Dom}(\psi)} [p \cdot u + \psi(p)]$$

The structure of the Lax-Hopf formula (4) implies the following important property, known as *inf-morphism* property [1].

Proposition 2 (Inf-morphism property) *Let the value condition* $\mathbf{c}(\cdot,\cdot)$ *be minimum of a finite number of lower semicontinuous functions:*

$$\forall (t,x) \in [0, t_{\max}] \times [\xi, \chi], \;\; \mathbf{c}(t,x) := \min_{j \in J} \mathbf{c}_j(t,x)$$

The solution $\mathbf{M_c}(\cdot,\cdot)$ *associated with the above value condition can be decomposed [1], [9], [10] as:*

$$\forall (t,x) \in [0, t_{\max}] \times [\xi, \chi], \;\; \mathbf{M_c}(t,x) = \min_{j \in J} \mathbf{M}_{\mathbf{c}_j}(t,x)$$

The inf-morphism property is a consequence of the structure of the Lax-Hopf formula (as a minimization problem), and of the definition of the value conditions. Another important properties of solutions to Hamilton-Jacobi PDEs is known as *inf-convolution* [1].

In this report, we consider initial, upstream and downstream boundary, and internal conditions. For simplicity, we assume that these conditions are piecewise linear, and continuous, which can be physically interpreted as piecewise constant initial densities and piecewise constant boundary flows.

0.3 Affine initial, boundary and internal conditions

Multiple types of value conditions can be incorporated into a control or estimation problem. In the present article, we include initial, boundary and internal conditions. The initial and boundary conditions are typically measured (with some error) using fixed sensors, such as inductive loop detectors, magnetometers or traffic cameras. Similarly, the internal conditions are partially measured using probe vehicle trajectories. Additional types of traffic-generated constraints can be thought of, for example constraints on the internal density of traffic, or constraints on the travel-time (which differ from internal conditions), or even hybrid components [12], though these constraints are not investigated in the present report.

0.3.1 Definition of affine initial, boundary and internal conditions

The formal definition of initial, upstream, downstream and boundary conditions associated with the HJ PDE (2) is the subject of the following definition.

Definition 2 ***[Affine initial, boundary and internal conditions]*** *Let us define* $\mathbb{K} = \{0, \ldots, k_{\max}\}$*,* $\mathbb{N} = \{0, \ldots, n_{\max}\}$ *and* $\mathbb{M} = \{0, \ldots, m_{\max}\}$*. Let* T *and* X *represent a temporal and spatial discretization step respectively (though an uniform discretization is chosen for convenience, and is not required by the mathematical framework). For all* $k \in \mathbb{K}$*,* $n \in \mathbb{N}$ *and* $m \in \mathbb{M}$*, we define the following functions, respectively called initial, upstream, downstream (boundary) and internal conditions [13]:*

$$M_k(t,x) = \begin{cases} -\sum_{i=0}^{k-1} \rho(i) X \\ -\rho(k)(x - kX) & \text{if } \; t = 0 \\ & \text{and } \; x \in [kX, (k+1)X] \\ +\infty & \text{otherwise} \end{cases} \tag{5}$$

$$\gamma_n(t,x) = \begin{cases} \sum_{i=0}^{n-1} q_{\text{in}}(i)T & \\ +q_{\text{in}}(n)(t-nT) & \text{if } \ x=\xi \\ & \text{and } \ t \in [nT,(n+1)T] \\ +\infty & \text{otherwise} \end{cases} \quad (6)$$

$$\beta_n(t,x) = \begin{cases} \sum_{i=0}^{n-1} q_{\text{out}}(i)T & \\ +q_{\text{out}}(n)(t-nT) & \\ -\sum_{k=0}^{k_{max}} \rho(k)X & \text{if } \ x=\chi \\ & \text{and } t \in [nT,(n+1)T] \\ +\infty & \text{otherwise} \end{cases} \quad (7)$$

$$\mu_m(t,x) = \begin{cases} L_m + r_m(t - t_{\min}(m)) \\ (\text{if } \ x = x_{\min}(m) \\ +\frac{x_{\max}(m)-x_{\min}(m)}{t_{\max}(m)-t_{\min}(m)}(t-t_{\min}(m)) \\ \text{and } t \in [t_{\min}(m), t_{\max}(m)]) \\ +\infty \qquad \text{otherwise} \end{cases} \quad (8)$$

.

As stated in the previous section, the initial, boundary and internal conditions defined above are usually not known exactly. In particular, we do not know the exact values of the initial densities $\rho(\cdot)$, the boundary flows $q_{\text{in}}(\cdot)$ and $q_{\text{out}}(\cdot)$ (which are precisely the objective of the present work), as well as the coefficients L_m and r_m of the internal conditions. Some coefficients such as $\rho(\cdot)$, $q_{\text{in}}(\cdot)$ and $q_{\text{out}}(\cdot)$ can be known with some uncertainty using flow or traffic density sensors, but some coefficients such as L_m and r_m simply cannot be measured experimentally by any traffic sensor. All of these unknown variables will act as part of our decision variable for the Mixed Integer Linear Program (MILP) derived in Section 0.4. Note that the coefficients $x_{\min}(\cdot)$, $x_{\max}(\cdot)$, $t_{\min}(\cdot)$ and $t_{\max}(\cdot)$ are known with high accuracy since they are typically measured with a GPS, and will thus not be part of the problem's decision variable.

0.3.2 Analytical solutions to affine initial, boundary and internal conditions

Given the affine initial, upstream, downstream and internal conditions defined above, the corresponding solutions $\mathbf{M}_{M_k}(\cdot,\cdot)$, $\mathbf{M}_{\gamma_n}(\cdot,\cdot)$, $\mathbf{M}_{\beta_n}(\cdot,\cdot)$ and $\mathbf{M}_{\mu_m}(\cdot,\cdot)$ defined by the Lax-Hopf formula (4) can be computed explicitly as analytical solutions [23, 7]. These expressions, in the case of the fundamental triangular diagram, are described below. Their formulation was initially derived in ¸

$$
\mathbf{M}_{M_k}(t,x) = \begin{cases} +\infty & \text{if } x \leq kX + wt \\ & \text{or } x \geq (k+1)X + vt \\ -\sum_{i=0}^{k-1} \rho(i)X & \\ +\rho(k)(tv + kX - x) & \text{if } kX + tv \leq x \\ \quad\quad \text{and} & (k+1)X + tv \geq x \\ \quad\quad \text{and} & \rho(k) \leq \rho_c \\ -\sum_{i=0}^{k-1} \rho(i)X & \\ +\rho_c(tv + kX - x) & \text{if } kX + tv \geq x \\ \quad\quad \text{and} & kX + tw \leq x \\ \quad\quad \text{and} & \rho(k) \leq \rho_c \\ -\sum_{i=0}^{k-1} \rho(i)X & \\ +\rho(k)(tw + kX - x) & \\ -\rho_m tw & \text{if } kX + tw \leq x \\ \quad\quad \text{and} & (k+1)X + tw \geq x \\ \quad\quad \text{and} & \rho(k) \geq \rho_c \\ -\sum_{i=0}^{k} \rho(i)X & \\ \rho_c(tw + (k+1)X - x) & \\ -\rho_m tw & \text{if } (k+1)X + tv \geq x \\ \quad\quad \text{and} & (k+1)X + tw \leq x \\ \quad\quad \text{and} & \rho(k) \geq \rho_c \end{cases} \tag{9}
$$

$$
\mathbf{M}_{\gamma_n}(t,x) = \begin{cases} +\infty & \text{if } t \leq nT + \frac{x-\xi}{v} \\ \sum_{i=0}^{n-1} q_{\text{in}}(i)T & \\ +q_{\text{in}}(n)(t - \frac{x-\xi}{v} - nT) & \text{if } nT + \frac{x-\xi}{v} \leq t \\ & \text{and } t \leq (n+1)T \\ & + \frac{x-\xi}{v} \\ \sum_{i=0}^{n} q_{\text{in}}(i)T & \\ +\rho_c v(t - (n+1)T - \frac{x-\xi}{v}) & \text{otherwise} \end{cases} \tag{10}
$$

$$
\mathbf{M}_{\beta_n}(t,x) = \begin{cases} +\infty & \text{if } t \leq nT \\ & + \frac{x-\chi}{w} \\ -\sum_{k=0}^{k_{max}} \rho(k)X + \sum_{i=0}^{n-1} q_{\text{out}}(i)T & \\ +q_{\text{out}}(n)(t - \frac{x-\chi}{w} - nT) & \\ -\rho_m(x-\chi) & \text{if } nT \\ & + \frac{x-\chi}{w} \leq t \\ \quad\quad\quad\quad \text{and} & t \leq (n+1)T \\ & + \frac{x-\chi}{w} \\ -\sum_{k=0}^{k_{max}} \rho(k)X + \sum_{i=0}^{n} q_{\text{out}}(i)T & \\ +\rho_c v(t - (n+1)T - \frac{x-\chi}{v}) & \text{otherwise} \end{cases} \tag{11}
$$

$$\mathbf{M}_{\mu_m}(t,x)=\begin{cases} L_m + \\ r_m\left(t-\frac{x-x_{\min}(m)-v^{\text{meas}}(m)(t-t_{\min}(m))}{v-v^{\text{meas}}(m)}-t_{\min}(m)\right) \\ \text{ if } x\geq x_{\min}(m)+v^{\text{meas}}(m)(t-t_{\min}(m)) \\ \text{ and } x\geq x_{\max}(m)+v(t-t_{\max}(m)) \\ \text{ and } x\leq x_{\min}(m)+v(t-t_{\min}(m)) \\ L_m + \\ r_m\left(t-\frac{x-x_{\min}(m)-v^{\text{meas}}(m)(t-t_{\min}(m))}{w-v^{\text{meas}}(m)}-t_{\min}(m)\right) \\ +k_c(v-w)\frac{x-x_{\min}(m)-v^{\text{meas}}(m)(t-t_{\min}(m))}{w-v^{\text{meas}}(m)} \\ \text{ if } x\leq x_{\min}(m)+v^{\text{meas}}(m)(t-t_{\min}(m)) \\ \text{ and } x\leq x_{\max}(m)+w(t-t_{\max}(m)) \\ \text{ and } x\geq x_{\min}(m)+w(t-t_{\min}(m)) \\ L_m+r_m\left(t_{\max}(m)-t_{\min}(m)\right)+ \\ \left(t-t_{\max}(m)\right)k_c\left(v-\frac{x-x_{\max}(m)}{t-t_{\max}(m)}\right) \\ \text{ if } x\leq x_{\max}(m)+v(t-t_{\max}(m)) \\ \text{ and } x\geq x_{\max}(m)+w(t-t_{\max}(m)) \\ +\infty \text{ otherwise} \end{cases} \tag{12}$$

These analytical expressions of $\mathbf{M}_{M_k}(\cdot,\cdot)$, $\mathbf{M}_{\gamma_n}(\cdot,\cdot)$, $\mathbf{M}_{\beta_n}(\cdot,\cdot)$ and $\mathbf{M}_{\mu_m}(\cdot,\cdot)$ are critical: they allow one to compute the solution to the HJ PDE (2) semi-analytically, from the inf-morphism property. They also enable one to formulate the problem of reconstructing initial (or boundary conditions, as in the present case) as an optimization problem. The first step in this process is to define the constraints defined by the Hamilton-Jacobi PDE (2).

0.4 Constraints arising from model and measurement data

We consider a set of block boundary conditions $\mathbf{c}_j$ defined as in Section 0.3.1, with unknown coefficients. Let us call V the hyperrectangle of unknown coefficients (V is an hyperrectangle since all the coefficients are real numbers that are bounded in magnitude, and of definite sign). Our measurement data (from the data set) constraints the possible values of these coefficients. Such constraints are called *data constraints*. Similarly, the model compatibility conditions also constraint the possible values of the unknown coefficients. Such constraints are called *model constraints*, and are outlined in Section 0.4.1.

0.4.1 Model constraints

We now describe the constraints of the Hamilton-Jacobi PDE (2) using the analytical solutions defined above as well as the inf-morphism property. These constraints were initially derived in [11], from the compatibility of boundary conditions blocks and solutions.

Proposition 3 ***[Model constraints]*** *The constraints of the Hamilton-Jacobi PDE (2) can be expressed as the following finite set of convex inequality constraints:*

$$\begin{cases} \mathbf{M}_{M_k}(0,x_p) \geq M_p(0,x_p) & \forall (k,p) \in \mathbb{K}^2 \\ \mathbf{M}_{M_k}(pT,\chi) \geq \beta_p(pT,\chi) & \forall k \in \mathbb{K}, \forall p \in \mathbb{N} \\ \mathbf{M}_{M_k}(\frac{\chi - x_{k+1}}{v},\chi) \geq & \\ \beta_p(\frac{\chi - x_{k+1}}{v},\chi) & \forall k \in \mathbb{K}, \forall p \in \mathbb{N} \text{ s. t. } \\ & \frac{\chi - x_{k+1}}{v} \in [pT, \\ & (p+1)T] \\ \mathbf{M}_{M_k}(pT,\xi) \geq \gamma_p(pT,\xi) & \forall k \in \mathbb{K}, \forall p \in \mathbb{N} \\ \mathbf{M}_{M_k}(\frac{\xi - x_k}{w},\xi) \geq \gamma_p(\frac{\xi - x_k}{w},\xi) & \forall k \in \mathbb{K}, \forall p \in \mathbb{N} \text{ s. t. } \\ & \frac{\xi - x_k}{w} \in [pT,(p+1)T] \end{cases} \tag{13}$$

$$\begin{cases} \mathbf{M}_{M_k}(t_{\min}(m),x_{\min}(m)) \geq \mu_m(t_{\min}(m),x_{\min}(m)) \\ \qquad \forall k \in \mathbb{K}, \forall m \in \mathbb{M} \\ \mathbf{M}_{M_k}(t_{\max}(m),x_{\max}(m)) \geq \mu_m(t_{\max}(m),x_{\max}(m)) \\ \qquad \forall k \in \mathbb{K}, \forall m \in \mathbb{M} \\ \mathbf{M}_{M_k}(t_1(m,k),x_1(m,k)) \geq \mu_m(t_1(m,k),x_1(m,k)) \\ \qquad \forall k \in \mathbb{K}, \forall m \in \mathbb{M} \text{ s. t. } t_1(m,k) \in [t_{\min}(m);t_{\max}(m)] \\ \mathbf{M}_{M_k}(t_2(m,k),x_2(m,k)) \geq \mu_m(t_2(m,k),x_2(m,k)) \\ \qquad \forall k \in \mathbb{K}, \forall m \in \mathbb{M} \text{ s. t. } t_2(m,k) \in [t_{\min}(m);t_{\max}(m)] \\ \mathbf{M}_{M_k}(t_3(m,k),x_3(m,k)) \geq \mu_m(t_3(m,k),x_3(m,k)) \\ \qquad \forall k \in \mathbb{K}, \forall m \in \mathbb{M} \text{ s. t. } t_3(m,k) \in [t_{\min}(m);t_{\max}(m)] \\ \mathbf{M}_{M_k}(t_4(m,k),x_4(m,k)) \geq \mu_m(t_4(m,k),x_4(m,k)) \\ \qquad \forall k \in \mathbb{K}, \forall m \in \mathbb{M} \text{ s. t. } t_4(m,k) \in [t_{\min}(m);t_{\max}(m)] \end{cases} \tag{14}$$

$$\begin{cases} \mathbf{M}_{\gamma_n}(pT,\xi) \geq \gamma_p(pT,\xi) & \forall (n,p) \in \mathbb{N}^2 \\ \mathbf{M}_{\gamma_n}(pT,\chi) \geq \beta_p(pT,\chi) & \forall (n,p) \in \mathbb{N}^2 \\ \mathbf{M}_{\gamma_n}(nT + \frac{\chi - \xi}{v},\chi) \geq \beta_p(nT + \frac{\chi - \xi}{v},\chi) & \forall (n,p) \in \mathbb{N}^2 \text{ s. t. } nT + \\ & \frac{\chi - \xi}{v} \in [pT,(p+1)T] \end{cases} \tag{15}$$

$$\begin{cases} \mathbf{M}_{\gamma_n}(t_{\min}(m),x_{\min}(m)) \geq \mu_m(t_{\min}(m),x_{\min}(m)) \\ \qquad \forall n \in \mathbb{N}, \forall m \in \mathbb{M} \\ \mathbf{M}_{\gamma_n}(t_{\max}(m),x_{\max}(m)) \geq \mu_m(t_{\max}(m),x_{\max}(m)) \\ \qquad \forall n \in \mathbb{N}, \forall m \in \mathbb{M} \\ \mathbf{M}_{\gamma_n}(t_5(m,n),x_5(m,n)) \geq \mu_m(t_5(m,n),x_5(m,n)) \\ \qquad \forall n \in \mathbb{N}, \forall m \in \mathbb{M} \text{ s. t. } t_5(m,n) \in [t_{\min}(m);t_{\max}(m)] \end{cases} \tag{16}$$

$$\begin{cases} \mathbf{M}_{\beta_n}(pT,\xi) \geq \gamma_p(pT,\xi) & \forall (n,p) \in \mathbb{N}^2 \\ \mathbf{M}_{\beta_n}(nT + \frac{\xi - \chi}{w},\xi) \geq \gamma_p(nT + \frac{\xi - \chi}{w},\xi) & \forall (n,p) \in \mathbb{N}^2 \text{ s. t. } nT + \\ & \frac{\xi - \chi}{w} \in [pT,(p+1)T] \\ \mathbf{M}_{\beta_n}(pT,\chi) \geq \beta_p(pT,\chi) & \forall (n,p) \in \mathbb{N}^2 \end{cases} \tag{17}$$

$$\begin{cases} \mathbf{M}_{\beta_n}(t_{\min}(m), x_{\min}(m)) \geq \mu_m(t_{\min}(m), x_{\min}(m)) \\ \qquad\qquad\qquad\qquad \forall n \in \mathbb{N}, \forall m \in \mathbb{M} \\ \mathbf{M}_{\beta_n}(t_{\max}(m), x_{\max}(m)) \geq \mu_m(t_{\max}(m), x_{\max}(m)) \\ \qquad\qquad\qquad\qquad \forall n \in \mathbb{N}, \forall m \in \mathbb{M} \\ \mathbf{M}_{\beta_n}(t_6(m,n), x_6(m,n)) \geq \mu_m(t_6(m,n), x_6(m,n)) \\ \qquad \forall n \in \mathbb{N}, \forall m \in \mathbb{M} \text{ s. t. } t_6(m,n) \in [t_{\min}(m); t_{\max}(m)] \end{cases} \tag{18}$$

$$\begin{cases} \mathbf{M}_{\mu_m}(pT, \xi) \geq \gamma_p(pT, \xi) & \forall (m,p) \in \mathbb{M} \times \mathbb{N} \quad (vii)(a) \\ \mathbf{M}_{\mu_m}(t_7(m), \xi) \geq \gamma_p(t_7(m), \xi) & \forall (m,p) \in \mathbb{M} \times \mathbb{N} \text{ s. t.} \\ & t_7(m) \in [pT, (p+1)T] \quad (vii)(b) \\ \mathbf{M}_{\mu_m}(t_8(m), \xi) \geq \gamma_p(t_8(m), \xi) & \forall (m,p) \in \mathbb{M} \times \mathbb{N} \text{ s. t.} \\ & t_8(m) \in [pT, (p+1)T] \quad (vii)(c) \end{cases} \tag{19}$$

$$\begin{cases} \mathbf{M}_{\mu_m}(pT, \chi) \geq \beta_p(pT, \chi) & \forall (m,p) \in \mathbb{M} \times \mathbb{N} \quad (viii)(a) \\ \mathbf{M}_{\mu_m}(t_9(m), \chi) \geq \beta_p(t_9(m), \chi) & \forall (m,p) \in \mathbb{M} \times \mathbb{N} \text{ s. t.} \\ & t_9(m) \in [pT, (p+1)T] \quad (viii)(b) \\ \mathbf{M}_{\mu_m}(t_{10}(m), \chi) \geq \beta_p(t_{10}(m), \chi) & \forall (m,p) \in \mathbb{M} \times \mathbb{N} \text{ s. t.} \\ & t_{10}(m) \in [pT, (p+1)T] \quad (viii)(c) \end{cases} \tag{20}$$

$$\begin{cases} \mathbf{M}_{\mu_m}(t_{\min}(p), x_{\min}(p)) \geq \mu_p(t_{\min}(p), x_{\min}(p)) \\ \qquad\qquad\qquad\qquad \forall (m,p) \in \mathbb{M}^2 \quad (ix)(a) \\ \mathbf{M}_{\mu_m}(t_{\min}(p), x_{\max}(p)) \geq \mu_p(t_{\max}(p), x_{\max}(p)) \\ \qquad\qquad\qquad\qquad \forall (m,p) \in \mathbb{M}^2 \quad (ix)(b) \\ \mathbf{M}_{\mu_m}(t_{11}(m,p), x_{11}(m,p)) \geq \mu_p(t_{11}(m,p), x_{11}(m,p)) \\ \qquad \forall (m,p) \in \mathbb{M}^2 \text{ s. t. } t_{11}(m,p) \in [t_{\min}(p), t_{\max}(p)] \quad (ix)(c) \\ \mathbf{M}_{\mu_m}(t_{12}(m,p), x_{12}(m,p)) \geq \mu_p(t_{12}(m,p), x_{12}(m,p)) \\ \qquad \forall (m,p) \in \mathbb{M}^2 \text{ s. t. } t_{12}(m,p) \in [t_{\min}(p), t_{\max}(p)] \quad (ix)(d) \\ \mathbf{M}_{\mu_m}(t_{13}(m,p), x_{13}(m,p)) \geq \mu_p(t_{13}(m,p), x_{13}(m,p)) \\ \qquad \forall (m,p) \in \mathbb{M}^2 \text{ s. t. } t_{13}(m,p) \in [t_{\min}(p), t_{\max}(p)] \quad (ix)(e) \\ \mathbf{M}_{\mu_m}(t_{14}(m,p), x_{14}(m,p)) \geq \mu_p(t_{14}(m,p), x_{14}(m,p)) \\ \qquad \forall (m,p) \in \mathbb{M}^2 \text{ s. t. } t_{14}(m,p) \in [t_{\min}(p), t_{\max}(p)] \quad (ix)(f) \\ \mathbf{M}_{\mu_m}(t_{15}(m,p), x_{15}(m,p)) \geq \mu_p(t_{15}(m,p), x_{15}(m,p)) \\ \qquad \forall (m,p) \in \mathbb{M}^2 \text{ s. t. } t_{15}(m,p) \in [t_{\min}(p), t_{\max}(p)] \quad (ix)(g) \end{cases} \tag{21}$$

where the coefficients $t_1(m,k)$, $x_1(m,k)$, $t_2(m,k)$, $x_2(m,k)$, $t_3(m,k)$, $x_3(m,k)$, $t_4(m,k)$, $x_4(m,k)$, $t_5(m,n)$, $x_5(m,n)$, $t_6(m,n)$, $x_6(m,n)$, $t_7(m)$, $t_8(m)$, $t_9(m)$, $t_{10}(m)$, $t_{11}(m,p)$, $x_{11}(m,p)$, $t_{12}(m,p)$, $x_{12}(m,p)$, $t_{13}(m,p)$, $x_{13}(m,p)$, $t_{14}(m,p)$, $x_{14}(m,p)$, $t_{15}(m,p)$ and $x_{15}(m,p)$ are given by equations (22), (23) and (24) below:

$$\begin{cases} t_1(m,k) = \frac{x_{min}(m)-(k+1)X-v^{\text{meas}}(m)t_{min}(m)}{v-v^{\text{meas}}(m)} \\ x_1(m,k) = x_{\min}(m)+ \\ v^{\text{meas}}(m)\left(\frac{x_{min}(m)-(k+1)X-v^{\text{meas}}(m)t_{min}(m)}{v-v^{\text{meas}}(m)} - t_{\min}(m)\right) \\ t_2(m,k) = \frac{x_{min}(m)-kX-v^{\text{meas}}(m)t_{min}(m)}{w-v^{\text{meas}}(m)} \\ x_2(m,k) = x_{\min}(m)+ \\ v^{\text{meas}}(m)\left(\frac{x_{min}(m)-kX-v^{\text{meas}}(m)t_{min}(m)}{w-v^{\text{meas}}(m)} - t_{\min}(m)\right) \\ t_3(m,k) = \frac{x_{min}(m)-kX-v^{\text{meas}}(m)t_{min}(m)}{v-v^{\text{meas}}(m)} \\ x_3(m,k) = x_{\min}(m)+ \\ v^{\text{meas}}(m)\left(\frac{x_{min}(m)-kX-v^{\text{meas}}(m)t_{min}(m)}{v-v^{\text{meas}}(m)} - t_{\min}(m)\right) \\ t_4(m,k) = \frac{x_{min}(m)-(k+1)X-v^{\text{meas}}(m)t_{min}(m)}{w-v^{\text{meas}}(m)} \\ x_4(m,k) = x_{\min}(m)+ \\ v^{\text{meas}}(m)\left(\frac{x_{min}(m)-(k+1)X-v^{\text{meas}}(m)t_{min}(m)}{w-v^{\text{meas}}(m)} - t_{\min}(m)\right) \end{cases} \tag{22}$$

$$\begin{cases} t_5(m,n) = \frac{nTv-v^{\text{meas}}(m)t_{\min}(m)+x_{min}(m)-\xi}{v-v^{\text{meas}}(m)} \\ x_5(m,n) = x_{\min}(m)+ \\ v^{\text{meas}}(m)\left(\frac{nTv-v^{\text{meas}}(m)t_{\min}(m)+x_{\min}(m)-\xi}{v-v^{\text{meas}}(m)} - t_{\min}(m)\right) \\ t_6(m,n) = \frac{nTw-v^{\text{meas}}(m)t_{\min}(m)+x_{\min}(m)-\chi}{w-v^{\text{meas}}(m)} \\ x_6(m,n) = x_{\min}(m)+ \\ v^{\text{meas}}(m)\left(\frac{nTw-v^{\text{meas}}(m)t_{\min}(m)+x_{\min}(m)-\chi}{w-v^{\text{meas}}(m)} - t_{\min}(m)\right) \\ t_7(m) = \frac{\xi-x_{\min}(m)+wt_{\min}(m)}{w} \\ t_8(m) = \frac{\xi-x_{\max}(m)+wt_{\max}(m)}{w} \\ t_9(m) = \frac{\chi-x_{\min}(m)+vt_{\min}(m)}{v} \\ t_{10}(m) = \frac{\chi-x_{\max}(m)+vt_{\max}(m)}{v} \end{cases} \tag{23}$$

and

$$
\begin{cases}
t_{11}(m,p) = \frac{x_{\min}(m)-x_{\min}(p)+v^{\text{meas}}(p)t_{\min}(p)-v^{\text{meas}}(m)t_{\min}(m)}{v^{\text{meas}}(p)-v^{\text{meas}}(m)} \\
x_{11}(m,p) = x_{\min}(p) + v^{\text{meas}}(p)\Big(- t_{\min}(p)+ \\
\frac{x_{\min}(m)-x_{\min}(p)+v^{\text{meas}}(p)t_{\min}(p)-v^{\text{meas}}(m)t_{\min}(m)}{v^{\text{meas}}(p)-v^{\text{meas}}(m)}\Big) \\
t_{12}(m,p) = \frac{x_{\max}(m)-x_{\min}(p)+v^{\text{meas}}(p)t_{\min}(p)-vt_{\max}(m)}{v^{\text{meas}}(p)-v} \\
x_{12}(m,p) = x_{\min}(p) + v^{\text{meas}}(p)\Big(- t_{\min}(p)+ \\
\frac{x_{\max}(m)-x_{\min}(p)+v^{\text{meas}}(p)t_{\min}(p)-vt_{\max}(m)}{v^{\text{meas}}(p)-v}\Big) \\
t_{13}(m,p) = \frac{x_{\min}(m)-x_{\min}(p)+v^{\text{meas}}(p)t_{\min}(p)-vt_{\min}(m)}{v^{\text{meas}}(p)-v} \\
x_{13}(m,p) = x_{\min}(p) + v^{\text{meas}}(p)\Big(- t_{\min}(p)+ \\
\frac{x_{\min}(m)-x_{\min}(p)+v^{\text{meas}}(p)t_{\min}(p)-vt_{\min}(m)}{v^{\text{meas}}(p)-v}\Big) \\
t_{14}(m,p) = \frac{x_{\max}(m)-x_{\min}(p)+v^{\text{meas}}(p)t_{\min}(p)-vt_{\max}(m)}{v^{\text{meas}}(p)-w} \\
x_{14}(m,p) = x_{\min}(p) + v^{\text{meas}}(p)\Big(- t_{\min}(p)+ \\
\frac{x_{\max}(m)-x_{\min}(p)+v^{\text{meas}}(p)t_{\min}(p)-vt_{\max}(m)}{v^{\text{meas}}(p)-w}\Big) \\
t_{15}(m,p) = \frac{x_{\min}(m)-x_{\min}(p)+v^{meas}(p)t_{\min}(p)-vt_{\min}(m)}{v^{\text{meas}}(p)-w} \\
x_{15}(m,p) = x_{\min}(p) + v^{\text{meas}}(p)\Big(- t_{\min}(p)+ \\
\frac{x_{\min}(m)-x_{\min}(p)+v^{meas}(p)t_{\min}(p)-vt_{\min}(m)}{v^{\text{meas}}(p)-w}\Big)
\end{cases}
\tag{24}
$$

Proof — First observe that $\forall(k,n) \in [0,k_{\max}]\times[0,n_{\max}], \text{Dom}(M_k)\cap\text{Dom}(\mathbf{M}_{\gamma_n}) = \emptyset$ and that $\forall(k,n) \in [0,k_{\max}]\times[0,n_{\max}], \text{Dom}(M_k)\cap\text{Dom}(\mathbf{M}_{\beta_n}) = \emptyset$. Thus, the constraints that ensure that a set of initial, upstream, downstream and internal conditions is defined in the strong sense (that is, the solution $\mathbf{M_c}$ satisfies $\mathbf{M_c}(\cdot,\cdot) = \mathbf{c}(\cdot,\cdot)$ on the domain $Dom(\mathbf{c})$) are:

$$
\begin{cases}
\mathbf{M}_{M_k}(0,x) \geq M_p(0,x) & \forall x \in [pX,(p+1)X], \forall(k,p) \in \mathbb{K}^2 \\
\mathbf{M}_{M_k}(t,\chi) \geq \beta_p(t,x_p) & \forall t \in [pT,(p+1)T], \forall(k,p) \in \mathbb{K}^2 \\
\mathbf{M}_{M_k}(t,\xi) \geq \gamma_p(t,\xi) & \forall t \in [pT,(p+1)T], \forall(k,p) \in \mathbb{K}^2 \\
\mathbf{M}_{M_k}(t,x) \geq \mu_m(t,x) & \forall t \in [t_{\min(m)},t_{\max(m)}], x = x_{\min(m)}+ \\
& v^{\text{meas}}(m)(t-t_{\min(m)})\forall(k,m) \in \mathbb{K}\times\mathbb{M} \\
\mathbf{M}_{\gamma_n}(t,\xi) \geq \gamma_p(t,\xi) & \forall t \in [pT,(p+1)T], \forall(n,p) \in \mathbb{N}^2 \\
\mathbf{M}_{\gamma_n}(t,\xi) \geq \beta_p(t,\xi) & \forall t \in [pT,(p+1)T], \forall(n,p) \in \mathbb{N}^2 \\
\mathbf{M}_{\gamma_n}(t,x) \geq \mu_m(t,x) & \forall t \in [t_{\min(m)},t_{\max(m)}], x = x_{\min(m)}+ \\
& v^{\text{meas}}(m)(t-t_{\min(m)})\forall(n,m) \in \mathbb{N}\times\mathbb{M} \\
\mathbf{M}_{\beta_n}(t,\xi) \geq \gamma_p(t,\xi) & \forall t \in [pT,(p+1)T], \forall(n,p) \in \mathbb{N}^2 \\
\mathbf{M}_{\beta_n}(t,\xi) \geq \beta_p(t,\xi) & \forall t \in [pT,(p+1)T], \forall(n,p) \in \mathbb{N}^2 \\
\mathbf{M}_{\beta_n}(t,x) \geq \mu_m(t,x) & \forall t \in [t_{\min(m)},t_{\max(m)}], x = x_{\min(m)}+ \\
& v^{\text{meas}}(m)(t-t_{\min(m)})\forall(n,m) \in \mathbb{N}\times\mathbb{M} \\
\mathbf{M}_{\mu_k}(t,x) \geq \mu_m(t,x) & \forall t \in [t_{\min(m)},t_{\max(m)}], x = x_{\min(m)}+ \\
& v^{\text{meas}}(m)(t-t_{\min(m)})\forall(k,m) \in \mathbb{M}\times\mathbb{M}
\end{cases}
\tag{25}
$$

The inequalities outlined in Proposition 3 result from the above constraints, which can be written as a finite set of inequalities owing the piecewise affine structure of the solutions (9), (10), (11) and (12).

An important property of the model inequality constraints is that they are piecewise linear convex in the unknown boundary condition coefficients. These constraints are also indendent of the measurement data associated with the problem: they only depend upon the relative positions of each of the initial, upstream, downstream and internal boundary condition blocks.

While these initial, boundary and internal condition coefficients are subject to the above model constraints, measurement data yields additional constraints on their possible values, by restricting for example the possible values that flow, density or labels can take. These constraints are independent from the model constraints, and are enforcing that the unknown initial and boundary condition coefficients are compatible with measurement data, allowing some degree of difference between measurement and decision variables. The difference allowed between the true value of the densities or flows and the corresponding measurements is a function of sensor performance characteristics. The internal conditions are not associated with any data constraints, as the data is embedded in their position information (in the space time domain), which is assumed to be perfect. In the remainder of this report, we assume that the data constrains are linear inequalities in the unknown coefficients of the initial and boundary conditions. This can for instance model situations in which the L_1 and L_∞ measurement errors of a sensor are upper bounded by some value. Note that a sensor for which the measurement error is bounded in the L_2 norm sense would yield convex quadratic data constraints, which would yield linear and quadratic convex constraints (QC). Constraints in the L_p norm sense yield in general convex constraints, though these constraints may not be easily implementable on standard convex optimization software. In practice, sensor performance is usually given in the L_∞ norm sense.

Thus, defining y as the decision variable: $y := \big(\rho_{ini}(1), \ldots, \rho_{ini}(k_m), q_{in}(1), \ldots, q_{in}(n_m), q_{out}(1), \ldots\ q_{out}(n_m)\big)$, a traffic estimation problem can be posed as the following optimization program, with linear inequality constraints:

$$\begin{array}{l} \text{Minimize } \ f(y) \\ \text{s. t.} \begin{cases} A_{model} y \leq b_{model} \\ C_{data} y \leq d_{data} \end{cases} \end{array} \tag{26}$$

In the above problem, $f(\cdot)$ is the objective function (used to select a solution among all solutions satisfying the model and measurement data constraints) for the traffic estimation under the compatibility constraints from the physical conditions and sensor measurements. See [13] for applications of this framework to various traffic flow estimation problems.

0.5 Boundary flow estimation on a single highway link

In the previous section, we showed that the constraints arising in general esitmation and control problems were piecewise linear convex (that is, linear) in some decision variable. In this section, we briefly extend this formulation to boundary estimation problems over highway sections.

The model constraints place physical limitations on the feasible values of the decision variable that arise from the configuration of the problem, together with the traffic flow model.

0.5.1 Data constraints

In the subsequent problems, we assume that initial conditions are given and cannot be controlled. For instance, in the examples below, the initial condition is fixed by setting prescribed measurements $\rho(m)^{meas}$, resulting in linear equality constraints in the decision variable described above.

Conversely, we assume that the inflows and outflows ($q_{in}(t)$ and $q_{out}(t)$) on the highway link are unknown, though, again, the proposed framework can also handle situations in which flows are partially or completely known, through the addition of data constraints.

0.5.2 Objective function

Several objective functions can be considered when solving the bondary estimation problem. These functions include: maximizing or minimizing the total average flow across both boundaries, minimizing the difference between the actual boundary flows and estimated target values (this minimization can be done in the L_1, L_2 or L_∞ sense, or simply having the most regular solution, that is the solution for which the variations of the flow in the L_1 norm sense are minimal (which is the objective function used as part of the compressed sensing problem).

In this preliminary example, we choose to minimize the boundary flows:

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{n_m}(q_{in}(i) + q_{out}(i)) \\ \text{s. t.} & \left\{ \begin{array}{l} A_{model}y \le b_{model} \\ A_{meas}y \le b_{meas} \end{array} \right. \end{array} \tag{27}$$

0.5.3 Compressive sensing

Compressive sensing is a promising way of solving underdetermined problems, such as the problem that we are solving in the present report. Compressed sensing uses the L_1 norm in optimization problems to impose sparsity in a solution, in that, the solution vector generated by a problem involving compressed sensing will be *sparse*, that is, have a large number of zero entries. Sparsity is particularly useful in boundary estimation problems, in which the flow is assumed not to vary much.

To regularize the solutions in the subsequent problems, we add an additional L_1 norm term in the objective. This term minimizes the difference between the consecutive boundary flows, in the form

$$\lambda \cdot \sum_{i=1}^{p-1}(|q_{in}(i+1) - q_{in}(i)| + |q_{out}(i+1) - q_{out}(i)|) \tag{28}$$

This constraint ensures that the consecutive values (in time) of the estimated boundary flows do not vary much in time.

0.5.4 Implementation

In the implementation of the above problem, we consider a single section of road, with constant model parameters (free flow speed, congestion speed, capacity) across both space and time.

The link is divided into 6 segments of equal length $X = 200$ meters. The initial density measurements on 6 segments are defined as 6 piece-wise affine constants that represent free flowing traffic. We also simulate 20 boundary conditions (10 boundary condition blocks associated with the upstream boundary, and 10 boundary condition blocks associated with the downstream boundary) with a granularity of T= 30 s. The boundary flows are variables in the present problem.

We solve the LP (27) using `IBM Ilog Cplex` on a `Macbook` operating `MacOS X`. As illustrated in Fig. 1, the optimal boundary flow estimation problem shows both the minimal and maximal flows that can be associated with the defined initial condition, while satisfying the constraints of both the model and the data. This type of problem involves a few tens of variables and a few hundreds of constraints, and was solved in a few milliseconds.

0.5.5 Computational complexity

Unlike gradient-based methods, the present formulation does not rely on a discretization of the PDE, and does not require that the solution is computed in all points of a computational grid. Let n represent the number of cells, and m represent the number of time steps. The number of decision variables required to solve such problems is at least $n \times m$, since the effects of the model are propagated through all cells, at all times. Our approach requires less variables: only $2 \times m$ variables are required for each link, assuming that the initial densities are fixed. If the considered links contain more than a few discretization steps, the present method can significantly improve the performance of boundary flow estimation in comparison with respect to classical methods.

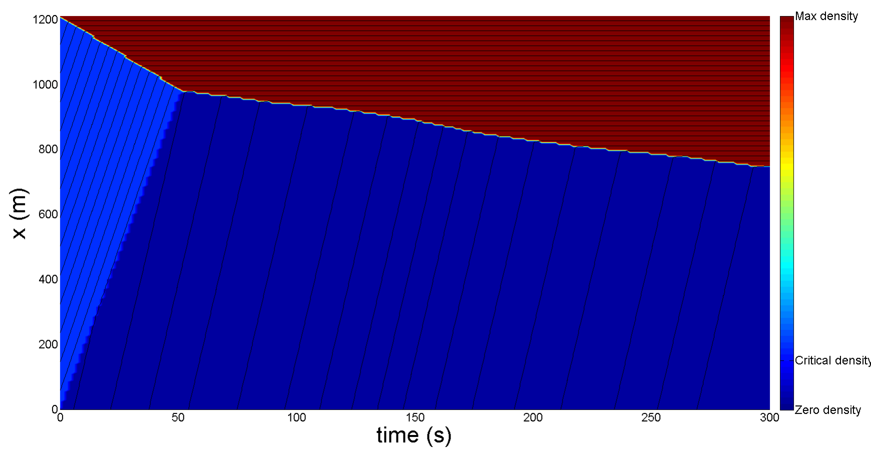
0.6 Generalization to Highway Networks

In the subsequent sections, we present a preliminary extension to boundary estimation in a highway network. In the remainder of this report, to avoid confusion with the terms q_{in}, q_{out} when modeling the junctions, we replace them with q^{us}, q^{ds} to denote upstream and downstream flows on links.

0.6.1 Network model

The highway network is modeled as a directed graph consisting of vertices $v \in \mathcal{V}$ and edges $e \in \mathcal{E}$. Each edge represents a link on the mainstream highway with a length L_e and a set of physical parameters (e.g. the speed limit v_{max}, the number of lanes $W_e \in \mathbb{N}^+$). On-ramps $on \in \mathcal{ON}$ are defined as a special edge with length $L_{on} \to 0$, and a set of parameters (e.g. v_{max}, a number of lanes $W_{on} \in \{0\} \cup \mathbb{N}^+$), and a direction to a vertex v. Off-ramp $off \in \mathcal{OFF}$ are defined similarly with $W_{off} \in \{0\} \cup \mathbb{N}^+$ and a direction out from a vertex v. A junction $j \in \mathcal{J}$ is defined as $\mathcal{J}_j := (v_j, \mathcal{I}_j, \mathcal{O}_j, on_j, off_j)$ and consists of a vertex $v_j \in \mathcal{V}$, a set of incoming edges $e_{in} \in \mathcal{I}_j$, a set of outgoing edges $e_{out} \in \mathcal{O}_j$, an on-ramp $on_j \in \mathcal{ON}$ and an off-ramp $off_j \in \mathcal{OFF}$. Note that this framework allows us a great deal of flexibility in defining arbitrary highway networks: if no on-ramp or off-ramp is present on a given section, we can set the number of lanes corresponding to the on- or off-ramp to zero, and eliminate it. An example of network layout is illustrated in Fig. 2.

To generalize the boundary estimation framework across junctions, we make the following assumptions:



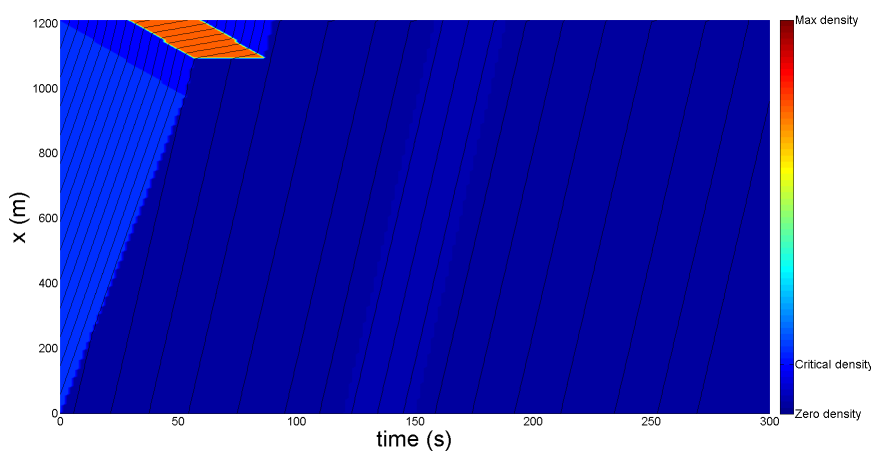


Figure 1: **Boundary flow estimation on a single link**: In this example, the initial condition is fixed. We compute the solutions respectively associated with the minimal and maximal possible values of the boundary flows (as defined in (27), modulo a sign change for the maximization of the flows). The top figure corresponds to the flow maximization scenario. The bottom figure represents the flow minimization scenario. In both figures, the compressed sensing term (28) was added to the objective to regularize the boundary flows.

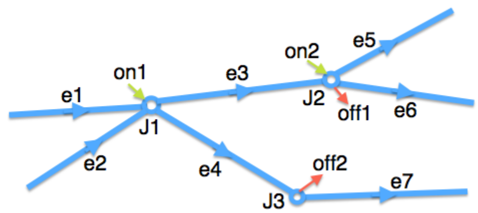


Figure 2: **Network graph definition**: Several highway links are connected by junctions. Each junction can have one on-ramp, one off-ramp, both, or none.

1. Junctions have no vehicle storage capacity.
2. At each junction $\mathcal{J}_j$, the incoming flows from $\mathcal{I}_j$ and on_j are routed to outgoing edges $\mathcal{O}_j$ and off_j according to a preference matrix.

With these assumptions, at a junction $j \in \mathcal{J}$ with incoming edges $e_{in} \in \mathcal{I}_j$, outgoing edges $e_{out} \in \mathcal{O}_j$, one on-ramp $on \in \mathcal{ON}_j$, and one off-ramp $off \in \mathcal{OFF}_j$, the variables of interest are related by equation (29) below:

$$\begin{bmatrix} \mathcal{A}_j & \mathcal{A}_j^{on} \\ \mathcal{B}_j & 0 \end{bmatrix} \begin{bmatrix} q^{ds} \\ q_j^{on} \end{bmatrix} = \begin{bmatrix} q^{us} \\ q_j^{off} \end{bmatrix} \tag{29}$$

Column vectors q^{us} and q^{ds} contain $q^{us}_{e_{out}}(t, \xi_{e_{out}})$, $q^{ds}_{e_{in}}(t, \chi_{e_{in}})$ denoting the upstream and downstream flows on respective links.
The variables $q_j^{on}(t, \cdot)$ and $q_j^{off}(t, \cdot)$ are scalar values representing flows on on-ramps on and off-ramps off.
$\mathcal{A}_j$ is a $|\mathcal{O}_j| \times |\mathcal{I}_j|$ matrix, where each element $\alpha^j_{mn}(t)$ denoting the percentage of the incoming flow $q^{ds}_{e_{in}}(t, \chi_{e_{in}})$ at this junction routed to an outgoing link $e_{out} \in \mathcal{O}_j$.
$\mathcal{A}_j^{on}$ is a $|\mathcal{O}_j| \times 1$ matrix, where each element $\alpha^{on}_{mn}(t)$ denotes the percentage of the on-ramp flow $q_j^{on}(t, \cdot)$ routed to an outgoing link $e_{out} \in \mathcal{O}_j$.
$\mathcal{B}_j$ is a $1 \times |\mathcal{I}_j|$ matrix, where each element $\beta^{off}_n(t)$ denotes the percentage of the incoming flow $q^{ds}_{e_{in}}(t, \chi_{e_{in}})$ routed to the off-ramp $off \in \mathcal{OFF}$.

From the conservation of vehicles across the junction, the total outflow from the junction is equal to the total inflow. Therefore, the parameters in Equation (29) satisfy:

$$\sum_{e_{out} \in \mathcal{O}_j} \alpha^j_{e_{out}, e_{in}}(t) = 1 - \beta^{off}_{e_{in}}(t) \quad \forall j \in \mathcal{J} \tag{30}$$

Note that this linear formulation misses an important additional constraint: the inflows and outflows entering and exiting a junction are typically maximized at all times [16], which is not enforced in the present formulation. This maximization comes from the intents of the drivers to arrive to their destination as early as possible. Known as the *entropy condition*, this condition would result in the definition of additional Boolean variables to represent the possible evolution of flows across the junction, which would significantly affect the computational time of the problem. To maximize the performance, we impose a corresponding term in the objective function, which corresponds to the maximization of boundary flows at all junctions, and for all times. Since this objective contradicts the original objective (for example the L_1 regularization objective), these constraints are not strictly enforced, and the solution to the corresponding optimization problem is approximate.

0.6.2 Formulation of the network boundary estimation problem

The model and data constraints for the single link boundary estimation problem (27) still apply on each link of the highway network. The model for junctions in Equation (29) (30) can be incorporated into the linear program as equality constraints.

Let us consider a highway network with links $\mathcal{E}$ and junctions $\mathcal{J}$. We define a new decision variable y as

$$y := [\overbrace{\rho_1^{ini}(0,\cdot),\ldots,q_1^{us}(t,\xi_{e_1}),\ldots,q_1^{ds}(t,\chi_{e_1})}^{\text{Link 1}},\ldots\overbrace{\rho_n^{ini}(0,\cdot),\ldots}^{\text{Link n}}, \\ q_1^{on}(t,\cdot),\ldots,q_m^{on}(t,\cdot)]$$

The variables $q_1^{on}(t,\cdot),\ldots,q_m^{on}(t,\cdot)$ can be regarded as the terms optimized in the present formulation. By defining a certain objective function $f(\cdot)$, the traffic boundary estimation problem can be posed as an optimization problem in which the constraints of the model and of the data are linear.

$$\begin{array}{ll} \text{Minimize} & f(y) \\ \text{s. t.} & \begin{cases} A_{model}y \leq b_{model} \\ C_{data}y = d_{data} \\ E_{conj}y = f_{conj} \end{cases} \end{array} \tag{31}$$

0.7 Application of the boundary estimation framework on highway networks

We now present a simulated application of the control framework on a toy highway network example (consisting in a diverge of two roads) to demonstrate its applicability to real-life problems. We choose the same objective function as previously, though, different weights could be attributed to the flows going through the different links. Figure 3 illustrates (as previously) the two extreme examples of flow minimization and maximization across the junction, given initial condition data.

In the above experiments, we estimate the values of the flows at the boundaries of the network from initial condition data only, though other type of data (such as the internal condition data mentioned earlier) could also be used as part of the estimation problem. One difficulty associated with internal condition data is the fact that such data requires the definition of additional Boolean variables in the optimization framework defined above, since Barron-Jensen/Frankowska solutions to Hamilton-Jacobi equations are discontinuous in general, and imposing continuous solutions requires the definition of such Boolean variables [6]. This can cause the problem to become significantly harder to solve, specifically if one wants to solve large scale problems.

Specifically, any type of data that is not defined at the boundaries of the computational domains associated with each link (i.e. either at time zero, or at the upstream and downstream boundary of each link) will require the definition of boolean variables, irrespective of the type of data (density, flow, or velocity). One of the remaining challenges associated with boundary condition estimation (using

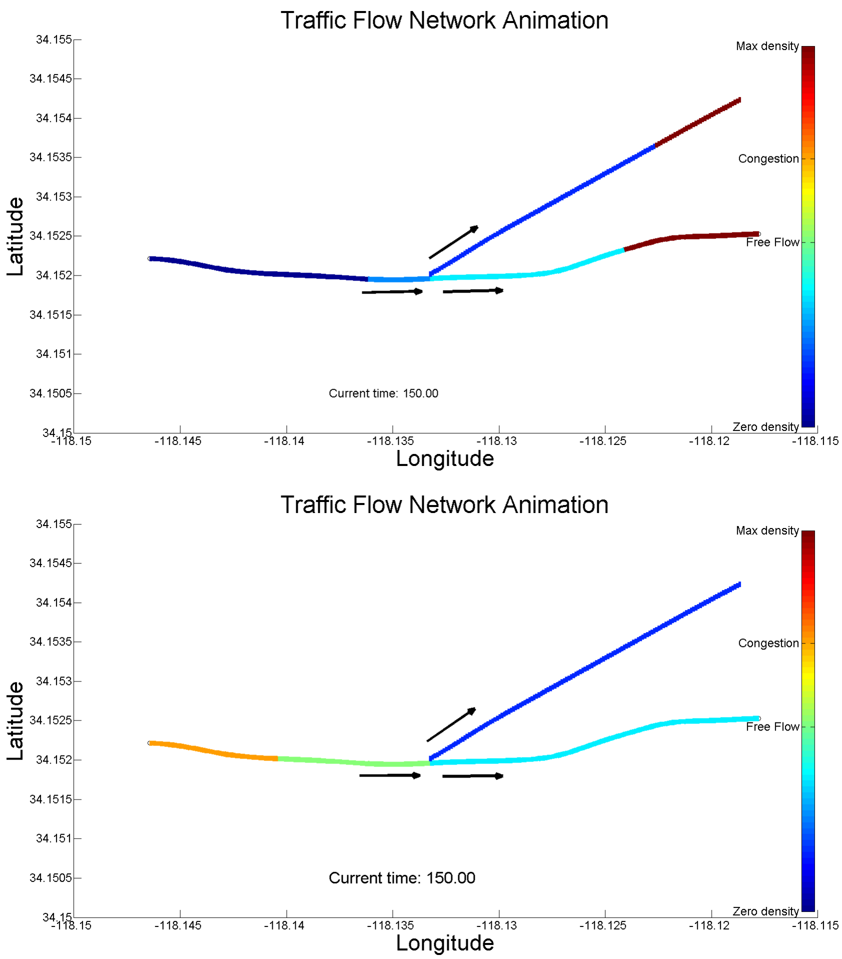


Figure 3: **Boundary flow estimation on an example road network**: In this example, the initial condition is fixed in all cases. We compute the solutions respectively associated with the minimal and maximal possible values of the boundary flows (as defined in (27), both across the junction and at the boundaries of the network. The top figure corresponds to the flow minimization scenario. The bottom figure represents the flow maximization scenario. In both figures, the compressed sensing term (28) was added to the objective to regularize the boundary flows.

this model-based framework) is to determine how much data can be added to large scale problems, and where should the data be added in priority to keep a tractable problem, yet get a relatively accurate approximate solution to the original problem.

0.8 Conclusion

This report proposes a new framework for boundary conditions estimation on transportation networks modeled by the Lighthill Whitham Richards partial differential equation. We first present an equivalent formulation of the problem based on a Hamilton Jacobi equation. The Lax-Hopf formula and Inf-morphism property of the solutions to the Hamilton Jacobi equation enable us to derive the constraints from the model and the measurement as a set of linear equalities in some decision variable. This enables us to pose the problem of single link boundary estimation as an optimization program with linear inequalities, most often a LP. The method can be extended to general transportation networks by modeling the propagation of traffic in junctions as linear equality and inequality constraints. This enables us to again pose the general problem of boundary flow estimation on networks as a MILP.

By leveraging the intrinsic properties of the LWR PDE, the proposed framework allows one to integrate arbitrary types of measurement data into a boundary flow estimation problem (which is a first step towards solving OD matrices estimation problems.

Future work will focus on the integration of the proposed framework into large scale OD estimation problems. The main issues are computational, but also related to the integration of human factors (human decisions, dynamic routing and dynamic traffic assignment) into the estimation problems. Indeed, changes in traffic patterns can also influence the route taken by users, through for example *demand elasticity*. The integration of these factors in a model-based flow estimation framework is still and open problem to date.

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