1. Report No. SWUTC/09/0-5881-1	2. Government Accession No.	3. Recipient's Catalog No.	
4. Title and Subtitle Network-based Decision Support Tool for Toll Roads		5. Report Date August 2009	
		6. Performing Organization Code	
7. Author(s) Ivan Damnjanovic, S.Travis Waller, Nevena Vajdic, David R. Suescún		8. Performing Organization Report No. 0-5581-1	
9. Performing Organization Name and Address Texas Transportation Institute The Texas A&M University System College Station, TX 77843-3135 Center for Transportation Research The University of Texas at Austin 3208 Red River, Suite 200 Austin, TX 78705-2650		10. Work Unit No.	
		11. Contract or Grant No. 0-5881	
12. Sponsoring Agency Name and Address Southwest Region University Transportation Center Texas Transportation Institute Texas A&M University System College Station, TX 77843-3135		13. Type of Report and Period Covered Technical Report September 2007-August 2009	
		14. Sponsoring Agency Code	
15. Supplementary Notes Project performed in cooperation with the Texas Department of Transportation and the Federal Highway Administration.			
16. Abstract Development, delivery, and operation of public infrastructure are becoming increasingly dependent on participation of the private sector. While revenue generating projects, such as toll roads, were traditionally developed and funded from the public sources, in recent years, as the public demand for new projects have exceeded the ability of the public sector to deliver them, the private investors have started to fulfill the gap between the needed and the available infrastructure. The objective of this research was to develop a network-based method that allows an assessment of the effect of the public sector's decisions regarding network improvements on the financial value of toll road projects.			
17. Key Words Toll Roads, Network Improvements, Feeder and Competing Links and Routes, Financial Feasibility		18. Distribution Statement No restrictions. This document is available to the public through the National Technical Information Service, Springfield, Virginia 22161; www.ntis.gov	
19. Security Classif. (of report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of pages 109	22. Price

Form DOT F 1700.7 (8-72)   **Reproduction of completed page authorized**

Network-Based Decision Support Tool for Toll Roads

Ivan Damnjanovic
S. Travis Waller
Nevena Vajdic
David R. Suescún

Texas Transportation Institute
Texas A&M University System
College Station, TX 77843-3135

Center for Transportation Research
University of Texas at Austin
3208 Red River, Suite 200
Austin, TX 78705-2650

August 2009

Disclaimer

The contents of this report reflect the views of the authors, who are responsible for the facts and the accuracy of the data presented herein. The contents do not necessarily reflect the official view or policies of the Federal Highway Administration or the Texas Department of Transportation (TxDOT). This report does not constitute a standard, specification, or regulation. This document is disseminated under the sponsorship of the Department of Transportation, University Transportation Centers Program, in the interest of information exchange.

Acknowledgments

Table of Contents

List of Figures

List of Tables

1. PROJECT OVERVIEW

1.1 Introduction

Improvements in the transportation system and maintenance of the existing facilities have traditionally been funded through state and federal budget appropriations and motor-vehicle fuel taxes. However, conventional funding mechanisms are no longer able to keep pace with the demand for maintaining the existing facilities, even without considering projects that add new capacity. In such settings, revenue-generating roadway projects, such as toll roads, have the potential to complement current project procurement practices while lessening the pressure on the public finances.

In such a framework, planning and valuation of a toll road project is inherently tied to the analysis of the supporting network. Regardless of the type of context-specific value measures, such as financial sustainability, system congestion or others, the actual contribution of the project to adopted value measures is always dependent on the network topology and aspects such as the capacities and conditions on competing and feeder routes and links. By considering the effects of actions on competing links and routes versus the effects of improvement actions on feeder links and routes, planning agencies gain tools to judge which network improvements will add the most value to the existing and planned toll road projects.

This research develops a strategic network-based evaluation framework that would allow TxDOT to quantify the impacts of its decisions about improvements in the transportation network on the value of existing and/or planned toll road projects. With such estimates, TxDOT gains tools to judge the financial value of both the supporting network and network improvement actions in the context of toll road projects.

1.2 Background

The public need for improved and expanded transportation networks increases constantly, thus placing higher pressure on the public sector for available funds needed for network improvements. In such a context, the public sector is currently looking at alternative methods for financing projects in an effort to bridge the gap between the existing and the needed funds.

Although the idea and concept of Public-Private Partnerships (PPPs) is not new, the advanced version of this method has served as an alternative for project delivery in recent years.

Project finance is one type of PPPs where the private sector and the lenders consider a set of forecasted revenues for their return on the investment and debt-service assessment (Yescombe, 2002). However, when arrangements between the public sector, the private sector, and lenders are negotiated, the revenue can only be forecasted and the uncertainty in its future outcome makes the revenue the main driver of the risk assessment.

Many strategies have been developed over the years to reduce and mitigate this risk. One of the strategies is a non-competing clause in the toll road agreements that constraints the public sector from improving the existing road network within some distance from the road of interest (Ortiz and Buxbaum, 2008). Otherwise, the public sector must pay the resulting revenue shortfall to the private investor. Other strategies, such as real options can be used as well, which can reduce risk exposure for all project participants (Nevitt and Fabozzi, 1995).

Since infrastructure projects with usage risk, such as toll roads, usually have high leverage (e.g. debt to equity ratio of 80:20) (Yescombe, 2002), the investor needs to secure a large amount of debt. In this regard, there are two main sources of debt financing: commercial banks and bonds. Since banks are flexible when it comes to renegotiating the loan when the borrower experiences debt servicing problems, these loans are more appropriate for the early phases of the project development, such as the construction and the start-up phase. During the project's operational phase, bonds are more suitable as the instrument for raising debt, because the project has established a certain level of continuous operation and some trends in cash inflows can be assessed at the time.

In either case, the risk assessment on the lender's side focuses again on the forecasted revenue of the project. The credit worthiness of the project is estimated solely on the future (and uncertain) revenue stream. It is then critical for both the borrower and the lender to price this risk properly. Over the years, many regulations were established with the intention of standardizing main principles in risk assessment and risk pricing for different financial agreements between lenders and borrowers. Nevertheless, due to the complex and specific nature of the PPP agreements, some regulations deal with the project finance agreements in general, leaving the risk assessment and risk pricing to the judgment of experts, as in the case of regulations for the banking business (Gatti *et al.*, 2007).

In this context, public agencies, private investors, and financial institutions are putting a great effort toward the understanding and development of PPP characteristics and financial models. This is being accomplished either through research efforts or data gathering from established road concessions throughout the world in an attempt to learn from previous experiences. As part of this process, this research aims at understanding the effects that decisions made by the public sector regarding improvements of the surrounding network will have on the financial feasibility of a toll road project. This is particularly relevant, because some public policies are part of the long-term planning process and their implementation can affect the project's performance under the PPP scheme either positively or negatively, thus potentially adding to or detracting from the value of the project.

Consequently, this research considers three financial instruments commonly used in PPPs: real options, bonds, and bank loans. Real options provide flexibility to the public or the private sector to reduce and mitigate the revenue risk in the negotiation phase of the project. Loans and bonds are widely used for raising debt, and negotiated between private parties which operate the project and financial institutions. This research addresses the different models used for risk assessment and risk pricing for these financial instruments in detail.

1.3 Problem Statement

Many questions arise between project participants in the negotiation phase of a toll road agreement. In order to ensure a successful communication between those participating in the negotiation phase, it is important to identify and quantify available information related to the project. This identification and quantification should also include information about changes in the network structure due to network improvement actions. Since network improvements are part of the long-term planning process in the public agencies, and therefore will be implemented some time in the future, they are likely to change the distribution of traffic flows in the network. In such a context, the relevant question is: *how will these network improvements affect the revenue on the toll road and the project's financial feasibility?* The current project proposes an appropriate modeling framework to answer this question.

When a new toll road is integrated into an existing transportation network, the travel patterns will inevitably change. Travelers will have new route options including full or partial use of the toll facility, or to avoid it completely. Because toll facilities depend directly on

payments from users to pay back the debt from construction costs and to accrue revenue, it is essential to consider how route choice decisions are made. Several recent analytical studies have examined the combined route choice and departure time decisions of drivers in the one origin, one destination, two-route network framework, however no research that we know of considers how changes in the existing network structure can affect (help or hurt) a new toll road or other existing toll road projects.

To ensure that the demand will support the debt-servicing obligation, toll road developers (public or private) need to consider the revenue risks through the consideration of the strategic position of the tolled links in the surrounding network. While economic development and population growth determine the increase in the total demand represented with an origin-destination (O-D) matrix, the actual distribution of the traffic is determined by the network structure and the condition on the network links. Furthermore, transportation networks are not static systems as they change and evolve over time. This dynamic redistribution of demand on the network links can be due, for instance, to network improvement actions. This observation is particularly relevant in the process of rating project debt, as rating agencies examine the effects of these dynamic changes on a project's ability to repay its debt.

Valuation of a toll road project is inherently tied to the analysis of the supporting transportation network, and as expected, the capacity and condition of feeder and competing routes and links in the supporting network can significantly determine the project's performance. In such settings, certain aspects of long-term transportation planning (such as quantification of link correlation, risk, and long-term uncertainty), which were not as critical to the traditional analysis, become absolutely paramount.

This research, therefore, develops a strategic network-based evaluation framework that would allow TxDOT to quantify the impacts of its decisions about improvements in the transportation network on the value of existing and/or planned toll road projects. There are several areas of contributions of this research: 1) Contract negotiation: the developed model can be employed in negotiating concession and risk allocation agreements, 2) Risk assessment: the model can be used to link transportation engineering and planning decisions and credit risk measures, and 3) Regulation: the model can be employed to aid financial regulators to understand the project's exposure to externalities resulting from changes in the structure of the transportation network.

1.4 Organization of the Report

This report is organized as follows. This chapter introduces and motivates the current research. The second chapter reviews relevant literature on financial, economic, and transportation engineering issues relevant to toll road projects. It also describes the role of PPPs in the procurement of toll roads and discusses issues that affect a project's exposure to risk. The third chapter explains the research methodology and describes the steps involved in the network-based evaluation framework for toll roads. The Fourth chapter presents detailed mathematical formulations for all the steps in the proposed methodology. Finally, the fifth chapter concludes this work.

2. LITERATURE REVIEW

2.1 Public Private Partnerships in the Transport Sector

Public-private partnerships (PPPs) are arrangements between the public and the private sector for developing or delivering public infrastructure or a public service (Hardcastle and Boothroyd, 2003). Project finance is one type of PPPs with non-recourse or limited recourse finance principle (Yescombe, 2002). While non-recourse finance refers to an agreement where the investors do not provide any guarantees when using debt to finance the project, limited recourse financing corresponds to any arrangement where the investors provide only a limited guarantee. Nevitt and Fabozzi (1995) explain that the final goal in project finance is an arrangement which will be beneficial for the investors without affecting their credit status. This is achieved through the above mentioned non-recourse financing.

In a report prepared by the World Bank and the Japanese Ministry of Construction (1999), 18 different countries are analyzed for the purpose of reviewing recent toll road experiences. It is noted that, private sector participation is significantly present in all the studied countries, even in those that traditionally had toll-free roads. In the report, strategic network planning is identified as one of key issues for the successful implementation of PPP agreements. In countries where entities involved in toll road programs were isolatedly established from the general network expansion planning programs, there were coordination and information exchange problems. It was also identified that the public transportation sector and the government faced planning and institutional issues, legal and regulatory issues, and concession contract issues.

Alexander, Estache, and Oliveri (2000) analyze different methodological problems that regulatory agencies and policy makers face in the transport sector. Their study reviews concessioner contracts and price regulations in order to establish the link between the authorities' impact and the market risk associated with transportation projects.

In their analysis of the communication in PPPs, Edwards and Bowen (2003) point out that the transfer of information between two parties is successful if the information is meaningful to both of them. Different perceptions of risks for the different involved parties place a special emphasis on risk communication in PPP agreements.

2.2 Risk in PPP Projects

Making choices under risk and uncertainty is a problem fundamental to many domains of scientific inquiry. In general, risk can be defined as the anticipation of events that can cause distress, loss, or damage. The consequences are expressed numerically and the likelihoods of occurrence are expressed as probabilities. In project finance, the assessment of the likelihood and the effects of such events is typically done by using a comprehensive procedure that involves identification, assessment, quantification, and allocation of all project risks, but also includes contingency planning for the risks that depend on some unobservable processes. With the increased participation of the private sector in the development and operation of transportation projects and networks, the problem of quantifying risks and opportunities becomes evermore pervasive.

For highly leveraged non-recourse projects, allocation of all identifiable risks is critical. Although risk management and transfer should be based on a party's competence to accept and manage risk, the commercial strength of the negotiators often defines risk allocation (Delmon 2005, Yescombe 2002). It is also important to note that, in addition to general risk allocation, there is a dynamic component of risk which propagates throughout a project's life (Delmon, 2005).

Some researchers have discussed the best approaches for categorizing risks in toll road projects. Li *et al*. (2001) present a conceptual model for risk allocation in PPP projects and categorize possible risk factors into three major groups: macro (ecological, political, economic, social), meso (project-engineering) and soft (micro level). Quiggin (2004) highlights the most important types of risks in PPPs and proposes a risk allocation rule based on the contractual party that is in best position to manage it. Yescombe (2002) defines general divisions of risk categories in project finance, suggesting that commercial risks, macroeconomic risks, and political risks are the major risk components of a project. Commercial risks are project specific risks; macroeconomic risks are external financial risks such as inflation risks; and political risks are country specific risks, resulting from the effects of governments' decisions on project performance measures. However, at the center of the task of risk assessment in project finance is the revenue risk. The revenue is the source for debt repayment and for creating returns to the investors.

2.3 Problems with Demand Forecasts

Analysis and forecast of traffic demand is an essential part of toll road development. Based on this forecast, a toll road project is designed in order to service estimated future traffic. Further, the financial structure of the project depends on this estimate. For example, the debt coverage of a project is calculated based on the estimated traffic demand and related toll road price (i.e. estimated toll road revenue). In practice, many toll road projects face financial problems due to traffic demand overestimation or underestimation. Flyvbjerg *et al*. (2006) present a statistical analysis of traffic forecasts for a sample of 210 infrastructure transportation projects and suggest that in about half of the sample of projects there were inaccuracies greater than 20% between the forecast and the actual realization of demand. For example, for the Dulles Greenway project, it was assumed that traffic demand would increase at a 14% rate for the first six years (Garvin and Cheah, 2004). However, the original estimate of 34,000 vehicles per day showed that the forecast was too optimistic; the actual average traffic per day was 11,500 vehicles in the first six months (Fishbein and Babbar, 1996). Due to traffic forecast errors, the government of Chile, for instance, provided three possible values for the traffic growth (i.e. 4, 4.5, and 5 percent) when offering concession agreements (Vassallo, 2006).

A committee formed to prepare a report on current practices of travel forecast models for the Transportation Research Board (2007) reports that the models used by Metropolitan Planning Organizations are in use for over fifty years and cannot address all new policy concerns. Smith (2006) considers the type of models used for estimating future toll road traffic and discusses factors that affect the performance of traffic forecasts, such as demographic and socioeconomic inputs, travel characteristics, value of time and willingness to pay, tolling culture, time choice modeling, and model validation.

In an attempt to reduce forecast uncertainty, Smith *et al*. (2004) develop a spreadsheet tool for preliminary estimation of revenues for toll road projects by considering three possible realizations (i.e. minimum, most likely, and maximum) for the traffic related input data (daily volume, growth rate, base toll diversion rate). Under some assumptions about toll collection, their analysis provides estimates of revenue, net present value of revenue, and ranking of alternatives based on a graphical-numerical description of traffic.

2.4 Models for Evaluating Toll Road Schemes

There is no standard method for toll road valuation. Aziz and Russell (2006) developed an economic model for infrastructure projects to provide a tool for project evaluation and risk analysis. Sanchez *et al*. (2005) developed a model for evaluating the feasibility of a toll road project and assessing the degree of financial risk in the pre-project planning phase.

Kriger *et al*. (2006) describe the most common methods used for toll road evaluation, which include: 1) assigning users to all roads – tolled and "free" – in a route choice model, 2) treating the choice to use toll roads explicitly within a mode split model, and 3) diversion curves to calculate the new facility's share of existing traffic. Route choice models require knowledge of users' value of time for converting monetary units into units of time. If a mode choice model is used, it must be iterated with a route choice model to achieve convergence. Diversion curve methods cannot consider traveler behavior and rely on a known elasticity of travel demand with respect to a toll rate.

2.5 Financial Instruments Used in PPP Agreements

In addition to the estimation and management of potential risks for a particular project, financial performance is an important issue for all stakeholders as they need to determine the potential of a project. The most common financial instruments used in PPPs are discussed below.

Bank Loans

Although projects can be financed from different sources, bank loans are widely used for financing infrastructure projects (Brealey *et al*., 1996; Nevitt and Fabozzi, 1995). Yescombe (2002) reports a growth of project finance bank loan obligations from $42,830 million in 1996 to $108,447 million in 2001 (Project Finance International). It is worth noticing that there are 20 major banks in the field of project finance with more than 70% participation in all project finance loans.

As the nature of the banking business is not simple and includes different types of agreements, the models that banks use for risk assessment are very complex. Different regulations were developed with the intention of standardizing those models. For example, the

new Basel II Accord (Basel Committee on Banking Supervision, 2004) is a regulatory document that regulates banks' exposure to loan risks. For each loan a bank offers, the bank needs to provide an adequate amount of capital to cover potential losses. The Basel Committee recognizes the probability of default, unexpected losses, expected losses, and losses given default as key parameters for credit risk analysis. However, there is a lack of quantitative models for the assessment of capital requirements for project finance loans under the new Basel II Accord requirements (Gatti et al., 2007).

Bonds

Bonds constitute one of possible sources that the private sector can use to raise additional funds during the project's operational phase (Yescombe, 2002). Nevitt and Fabozzi (1995) report that the use of bonds as a potential source for funding debt in infrastructure projects has increased and that the trend is expected to continue in the future. For example, in 1996 the total value of project finance bonds was $4,791 million, while this total reached $25,003 million in 2001 (Yescombe, 2002). Two types of bonds are typically used for financing infrastructure projects: revenue bonds and general obligation bonds. While general obligation bonds are secured by the government's power to repay obligations, revenue bonds are secured only by revenue from tolls or user charges.

In credit risk analysis, there are three main quantitative methods for pricing bonds: structural, reduced, and incomplete information approach (Giesecke, 2004). At the center of the credit risk assessment for bonds is the probability of default or probability of failure of the bond issuer to fulfill the financial agreement with the bond investors.

There are several approaches used in credit risk models. Crouhy *et al*. (2000) analyze and compare four credit risk models: the credit migration approach is based on the assessment of the probability of moving from one credit grade to another over some time horizon; the structural approach is based on the assessment of the asset value and the probability of default on debt service; the actuarial approach assumes that the probability of default, as the only parameter of interest, follows a Poisson process; and the CreditPortfolioView model addresses the probability of default as a function of macroeconomic variables.

Options

An option represents a contract between two parties, which grants the right to one party, but not the obligation, to buy or sell an asset for a pre-specified price (Trigeorgis, 1998). This right can be exercised at, before, or on a pres-specified date. If the option can be exercised before the end of the contract, it is called an American option; if it can be exercised only at the end of the contract, it is called a European option.

The theory of option pricing dates back to Merton (1973), who derived explicit formulas for pricing European call and put options and the options with a boundary condition (down-and-out). Rubinstein and Reiner (1991) and Rich (1994) developed pricing formulas for four types of European boundary options: down-and-out, down-and-in, up-and-out and up-and-in. Kunimoto and Ikeda (1992) extended the problem by developing a valuation formula for European options with curved boundaries. Geman and Yor (1996) used the Laplace transform for the derivation of a pricing formula based on the fundamental properties of Brownian motion. All these methods for option pricing are developed based on the price of an underlying asset, its volatility, and the exercise price as a function of the asset price.

Modeling of Financial Instruments in PPPs

Zhao *et al*. (2004) use a real options approach to decision making in build-operate-transfer (BOT) projects by taking into account traffic demand, land price, and highway deterioration uncertainties. Monte Carlo simulation is used to obtain a set of optimum decisions under the considered uncertainties.

Huang and Chou (2005) consider concurrently a minimum revenue guarantee and an option to abandon a project. The revenue guarantee is defined as the owner's obligation to cover a gap between a specified level of revenue and a possible shortfall of real revenue flow. The option to abandon a project is held by the concessionaire and defined under an investment option at the contract signing stage. The results show that both options address the same root cause of project failure, hence their values are correlated. In such settings, where multiple real options exist, a unified valuation model is needed to determine their combined value. Chiara (2006) develops two project valuation methods by combining Monte Carlo simulation and dynamic programming techniques for pricing revenue guarantee options.

To model uncertainty about the future revenue, Irwin (2003) and Brandao and Saraiva (2008) use the properties of geometric Brownian motion for the assessment of an option's value in infrastructure projects. The type of option analyzed in these models is a minimum traffic guarantee, which is modeled as a European option. Chiara and Garvin (2007) use two different methods for evaluating a minimum revenue guarantee: the multi-least square Monte Carlo method and the multi-exercised boundary method. Chow and Regan (2009) use a geometric Brownian motion assumption to model future travel demand as a key concept in real options analysis for managerial flexibility in network investments.

2.6 Transportation Network Modeling Literature

The following section discusses relevant literature on transportation network modeling as well as different economic issues related to pricing and private ownership and operation of transportation facilities.

Pricing

Traffic congestion represents the typical case of a negative externality which, from a theoretical perspective, could be internalized by implementing marginal social cost pricing. This corresponds to *first-best* regulation, in which individual users are charged a toll that equals the difference between their private cost and the total cost experienced by the link users due to the individual's decision to travel. In terms of network modeling, Beckmann *et al.* (1956) show that it is possible to achieve a system optimum pattern starting from user equilibrium, if one charges all links at the marginal cost, resulting in first-best regulation in which system performance is maximized. Dafermos (1973) and Smith (1979) extend the marginal social cost pricing concept to calculate the tolls on networks when the link costs are not separable. Depending on the objective of the toll setting problem, numerous models and solution algorithms exist to arrive at the optimal first-best link tolls (Hearn and Yildirim, 2001; Yildirim and Hearn, 2005).

In practice, there are technical, political, social, and institutional restrictions that require the consideration of *second-best* policies. Second-best regulation applies, for instance, when tolls can only be imposed on a subset of network links. In this direction, Lévy-Lambert (1968), Marchand (1968), and Verhoef *et al.* (1996) study a static model corresponding to the simplest network (i.e. two roads in parallel connecting an origin-destination pair)*, where only one of the

roads is subject to tolling. Those studies conclude that the magnitudes of the second-best tolls are less than those of the first-best tolls, and that the efficiency gains are less than those that can be reached under first-best regulation. It is also found that second-best regulation is effective to re-assign vehicle flows between the two routes.

In an effort to analyze larger networks, another approach to second-best tolling employs a bi-level optimization framework. Verhoef (2002) presents a model where a planner aims at finding the congestion tolls that maximize social welfare while the users choose their routes by minimizing their generalized travel cost. The optimal second-best congestion tolls for the subset of links being charged are found to depend on terms reflecting the costs and demand on other links, identifying the importance of taking into account the interactions in the road network. Zhang and Yang (2004) study a bi-level program similar to that studied by Verhoef (2002) to investigate an optimal *cordon toll* level as well as the optimal location of the cordon. Sumalee (2004), Shepherd and Sumalee (2004), and Sumalee, May and Shepherd (2005) also present bi-level formulations to investigate the problem of optimal toll levels and locations by employing a genetic algorithm-based approach. Although these kinds of works have advanced the theory, their approach to finding optimal toll levels and locations cannot yet be applied to actual complex networks (Santos, 2004). In addition, one must also notice that, given the intrinsic non-convexity of the bi-level optimization formulations, there is an inherent difficulty associated with ensuring that global solutions are found.

Other studies found in the literature, such as those presented by May and Milne (2000), Santos, Newbery, and Rojey (2000), and Santos (2004) study possible effects of second-best tolling on the spatial patterns of traffic flow and congestion by employing *network simulations*. Although those network simulation studies are of practical interest, they do not consider second-best optimal tolls, but a series of exogenously given toll levels instead.

So far, we have focused our attention on static models, which are commonly used due to their simplicity. Static models constitute a good approximation of traffic conditions that evolve slowly. Dynamic models of tolling require knowing how demand changes with space and time. Among the economic models, the most relevant work corresponds to the "*Bottleneck Model*," first proposed by Vickrey (1969), and refined by authors such as Braid (1989, 1996), and Arnott, De Palma, and Lindsey (1990, 1993, 1994). In these models, the tolls are set to reduce the development of queues and the costs associated with congestion. In the transportation science

field, dynamic road pricing has recently drawn increasing attention from the network research community. To support the planning, operation, and evaluation of various dynamic road pricing schemes, a user equilibrium *dynamic traffic assignment* (DTA) model is often used to predict path choices and the resulting network flow patters which can then be used to assess the impacts of proposed toll facilities (Mahmassani *et al*., 2005; Lu *et al*., 2006; Lu *et al*., 2007; Joksimovic, 2005).

Private Ownership and Operation

Analytical studies have also been conducted on various economic issues involving *privately owned highways*. Viton (1995) assesses the economic viability of private roads for the situation where a private toll road competes with a free-access road. He concludes that the private road is highly profitable under a range of assumptions about the mix of vehicle types and the costs of travel time, and presents a discussion of regulatory approaches to modify the impacts of simple profit-maximization.

Verhoef *et al*. (1996) examine two private ownership regimes: a regime in which one route is private and the other is free access, and a second regime in which one private firm controls both routes. Mills (1995) discusses the possibility of divergence between profit and welfare for a tolled link in a simple road network. He shows that a link that is profitable can nevertheless bring a negative welfare increment. Tsai and Chu (2003) analyze the regulation alternatives on private highway investment under a BOT scheme and their impacts on traffic flows, travel costs, toll, capacity and social welfare, and Yang and Woo (2000) examine the competitive Nash equilibrium by considering two toll roads provided and operated by two profit-maximizing private firms. De Palma and Lindsey (2000) study competition under various ownership regimes for private toll roads by looking at a model featuring one origin and one destination linked by two parallel routes that can differ in capacity and free-flow travel time. In their model, congestion takes the form of queuing and prospective travelers decide whether to drive, and if so on which route and at what time. Three private ownership regimes are considered: 1) a private road on one route and free access on the other, 2) a duopoly of private roads, and 3) a mixed duopoly with a private road competing with a public toll road. Private and public toll roads alike are found to generate substantially higher efficiency gains when tolls are varied over time to prevent queuing than when (time-invariant) tolling is adopted. Although

these analyses provide theoretical insights from an economic perspective, the conclusions are limited to very simple networks consisting of parallel or serial links.

In an effort to consider the ***impact of the network structure*** on the profitability and/or social welfare levels that could be reached under various ownership schemes, other researchers have developed approaches that can be applied to general networks. Yang and Meng (2000) and Yang *et al.* (2002) propose bi-level optimization formulations aimed at determining the optimal capacity and toll levels in the context of BOT projects under various market conditions. Subprasom *et al.* (2003) study the optimal selection of capacity and toll level in a BOT scheme under multiple uncertainties: future travel demands, cost estimates, and value of time. The results of their numerical experiments reveal the importance of considering multiple uncertainties when evaluating the feasibility of a BOT project. Subprasom and Chen (2007) evaluate the profit and welfare gain for both the public and private sector under different toll-capacity combinations. Chen *et al.* (2001) and Chen *et al.* (2003) study the impact of demand uncertainty on the performance of a highway BOT project by considering the maximization of the expected profit and the minimization of the variance of profit as the objectives in their mathematical programming formulations. Chen *et al.* (2007) extend the above framework and study the performance of a highway BOT project by considering three perspectives: the government, whose objective is to maximize social welfare; the private sector, whose objective is profit maximization; and the road users, whose objective is the minimization of the spatial inequity. These studies provide valuable insights into the selection of optimal roadway capacity and toll levels, and propose methodologies with the potential to be applied to real world networks.

Uncertainty in Modeling

While sophisticated methodologies for the quantification of uncertainty in traditional transportation planning are not yet common, it is clear that when considering large-scale network toll projects it is essential. One major uncertainty factor is travel demand. It is impossible to know the exact demand that will exist in the future, and when the network is altered (i.e. by adding a toll road) the uncertainty is exacerbated.

Studying the impact of ***demand uncertainty*** on the traffic assignment problem has received increasing attention in recent years. Waller *et al.* (2001) study the impact of long-term demand uncertainty and show that using a single fixed estimate of future demand can result in

significant underestimation of the future system performance. Numerous other works have focused on developing bi-level mathematical programming formulations and solution algorithms for the network design problem under uncertain demand (Karoonsoontawong and Waller, 2001; Ukkusuri *et al*. 2007; Waller and Ziliaskopoulos, 2007.).

Duthie *et al*. (2007) study the impact of **correlations** in the matrix of travel demands between each origin-destination pair on system performance and compare the efficiency of various sampling techniques to arrive at the expected future system performance. Lam *et al*. (1998) study the impact of demand uncertainty on toll revenue using Monte Carlo Simulation. The focus of their work is on determining a probability distribution for the future toll revenue and traffic flow when numerous input parameters like population and probable toll charges are assumed to follow a normal distribution. As mentioned before, Chen *et al*. (2003, 2007) study the problem of setting optimal tolls and capacity on a subset of links in a highway BOT project under demand uncertainty. Nagae and Akamatsu (2006) formulate the problem of choosing the optimal toll level from two discrete values as a stochastic singular control problem where the demand is assumed to vary following a stochastic differential equation. Clark and Watling (2005) study the impact of day-to-day variation in demand by analytically deriving the probability distribution of the total system travel time when demand is assumed to follow a poisson distribution. Shao *et al*. (2006) develop a multi class reliability-based stochastic equilibrium model where every user has a safety margin for arrival time. They assume the stochastic travel times to be caused by day-to-day demand variations. Gardner *et al*. (2007) propose a methodology to determine optimal and robust first-best tolls under uncertain demand.

Uncertainty in traffic assignment has also been studied by focusing on link capacity. Bell and Cassir (2000) present a comprehensive review of the various definitions of **capacity reliability**. Chen *et al*. (1999) define capacity reliability as the probability that a network can accommodate a certain demand at a given level of service. Lo and Tung (2003) deal with the problem of link capacities subject to stochastic degradation day-to-day, and define capacity reliability as the maximum flow that the network can carry given link capacity and travel time reliability constraints. Du and Nicholson (1997) propose a conventional equilibrium approach with variable demand to describe flows in a network with degradable link capacities. Unlike long-term demand, uncertainty about the system's capacity due to factors like incidents and work

zones is more important for short-term operational decisions than for planning decision years into the future.

Finally, it is worth mentioning that, uncertainty in ***user perception of travel time*** is typically dealt with using stochastic user equilibrium (SUE) methods. SUE allows for user classes with varying perceptions of travel time. The probability of choosing each route is determined, for example, by a stochastic loading model such as a logit model, the STOCH algorithm developed by Dial (1971), or the SAM probit model (Maher and Hughes, 1997). A mathematical program for SUE is given in Sheffi and Powell (1982). SUE approaches assume demand is known or elastic, but paths are chosen stochastically. A SUE assignment could be used with long-term demand uncertainty; however the difference in routing decisions will likely not impact planning policy or infrastructure choices.

3. RESEARCH METHODOLOGY

3.1 Research Framework

The research framework presented below was developed to allow an objective assessment of the effect of changes in the network structure on the value of the existing and new toll road projects. The main activities of this research focused primarily on four tasks:

- Identification and quantification of risk dependent value measures
- Identification of competing and feeder links and routes
- Quantification of network improvement actions
- Development of the decision support process

During the *Identification of risk dependent value measures* step, the research team first overviewed TxDOT's strategic goals: expand economic opportunity, increase the value of transportation assets, reduce congestion, enhance safety, and improve air quality.

Table 3.1: Strategic Goals, Focus Fields, and Performance Measures

TxDOT Strategic Goals	Focus Fields	Example Performance Measure
Expand Economic Opportunity	Toll Road Performance	Toll Revenue Operation and maintenance cost Return on investment (number of measures) Current bond rating
Increase Value of Transportation Assets		
Reduce Congestion	System Performance	Total System Travel Time
Enhance Safety	Safety	Annual number of crashes per lane mile
Improve Air Quality	Air Quality	Daily oxides of nitrogen (NOx), carbon monoxide (CO), and volatile organic compounds (VOC) emissions in grams per lane mile

After consideration of all the performance measures associated with these strategic goals (see Table 3-1), it was decided to keep the focus of the research on the revenue, as it is the main risk driver in the financial analysis of toll roads. In other words, the most relevant consideration of the impact of network improvement actions on the value of the toll road project was placed on pricing the revenue risk. Note that, actions that add to the value of the project correspond to those actions which will decrease the price of the risk. Also, note that, since a toll road is not a tradable asset itself, the worthiness/value of the project is estimated solely on the forecasted/expected cash inflows. Since these cash flows are forecasted and, hence, uncertain, the worthiness of the project will reflect this risk. The higher the risk, the less the project will be worth. The price of accepting and bearing the revenue risk will be high if the uncertainty about the future revenue is high. Thus, adding value to the project means decreasing this risk, or, in other words, decreasing the price of the revenue risk. Similarly, decreasing the value of the project means increasing the price of the revenue risk.

The *Identification of competing and feeder links* task provides a tool that fills a gap in the current practices of toll road analysis. While much of the current toll road research has focused on considering the effects of the network conditions and topology on route choice and network assignment, a related and equally important aspect of the network analysis is often neglected: the aspect of evaluating the strategic position of a toll road project in a larger transportation network. This aspect is particularly important in the process of securing sufficient funds for the construction and maintenance of a toll road project. In this research's context, competing routes/links are defined as routes/links whose presence detracts traffic from using the facility under consideration, while feeder routes/links correspond to those routes/links that help traffic access the facility. Within a traffic assignment framework, and based on these definitions, the proposed methodology first simulates variations in link capacity and then studies the resulting correlation patterns to identify the impact of capacity variations on link volumes and most likely route flows. Negative/positive correlations between the capacity of the link (or set of links) of interest and the traffic volumes on other links are used to identify competing/feeder links, while negative/positive correlations between the capacity of the link (or set of links) of interest and the most likely route flows on the routes connecting a given O-D pair are useful for identifying competing/feeder routes.

Quantification of network improvement actions was related to the assessment of the

20

actual impact that network improvement actions will have on the toll road. While identification of feeder and competing links reveals the role played by the network links (via the calculation of correlations) with respect to the toll road facility under consideration, the process of quantification determines the actual magnitude of the impact resulting from actual changes in the network structure. As discussed in the following section, this research addresses the quantification process through the estimation of elasticities of the flow or revenue on the toll road facility to changes in the capacity on a subset of links in the network.

Finally, the last component of this research, *development of the decision support process*, provides a modeling framework that allows an objective assessment of the effects of network improvement actions on the (financial) value of the toll road project. The following section presents a detailed explanation of this tool.

3.2 Evaluation Method

This research proposes a network-based evaluation framework, which subject to changes in the network structure initiated by the public sector, relates these changes to the financial instruments used in toll road agreements (Figure 3-1). As discussed below, several steps are proposed in order to develop a methodology that ensures an objective assessment of the effects of changes in the network structure on the value of a toll road project.

In the first step of the evaluation method, given the original network topology, its characteristics, and the demand for trips (i.e. Origin-Destination trip table), the traditional traffic assignment step from the four-step transportation planning model is used to determine the flow on all the network links[1].

---

[1] Although in all the applications presented here, this research team used its own computer code to execute the Frank-Wolfe algorithm and solve for the User Equilibrium flows resulting from deterministic traffic assignment, it is important to mention that there are several demand modeling packages in the market that can be used here (e.g. TransCAD (Caliper Corp. 2001) which is used by TxDOT's TP&P Division, EMME/2 (INRO, 2007), and VISUM (PTV Inc., 2007), just to name a few)

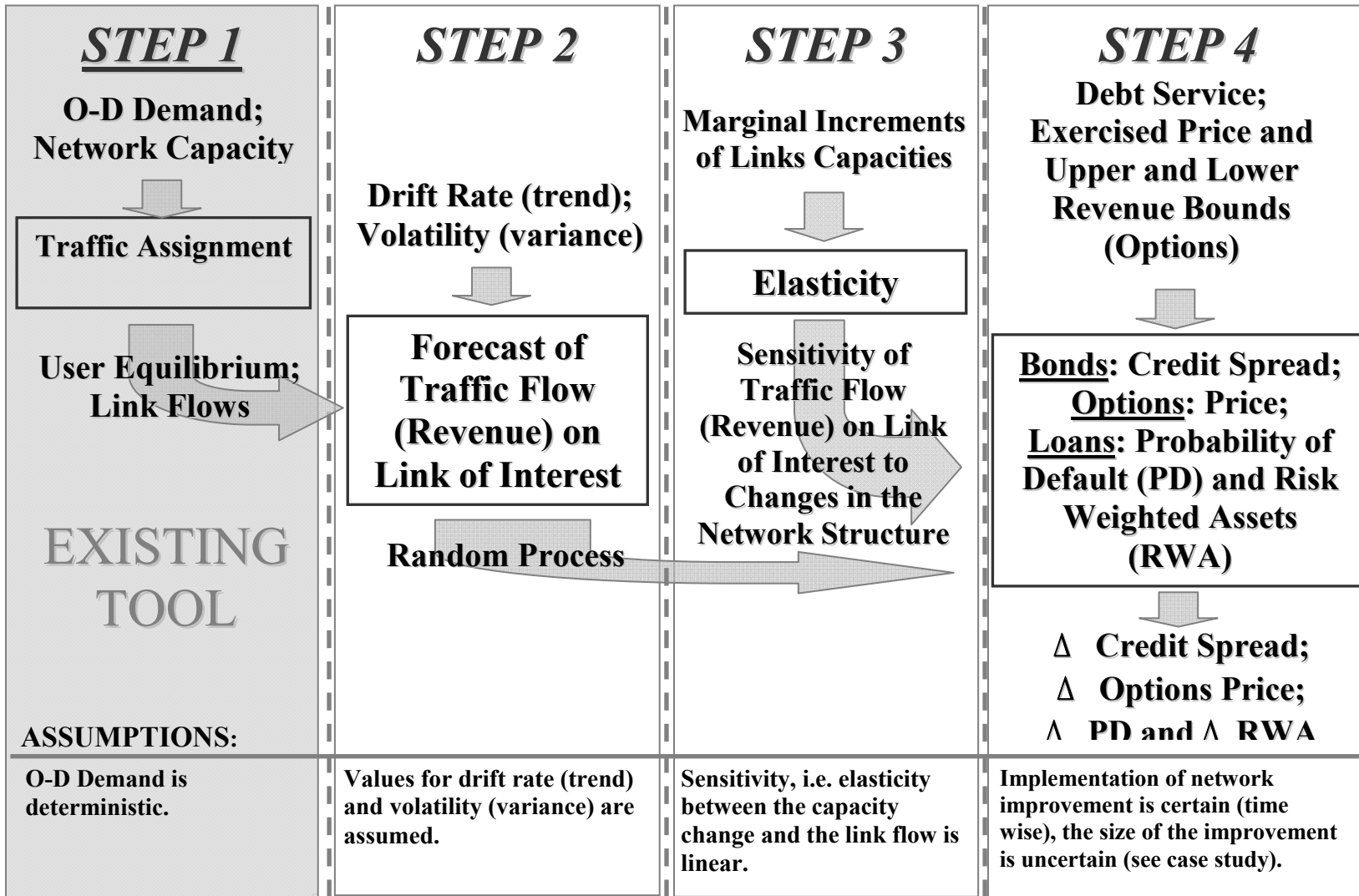| STEP 1	STEP 2	STEP 3	STEP 4
O-D Demand; Network Capacity		**Marginal Increments of Links Capacities**	**Debt Service; Exercised Price and Upper and Lower Revenue Bounds (Options)**
⇩	**Drift Rate (trend); Volatility (variance)**	⇩	
Traffic Assignment	⇩	**Elasticity**	⇩
	Forecast of Traffic Flow (Revenue) on Link of Interest	**Sensitivity of Traffic Flow (Revenue) on Link of Interest to Changes in the Network Structure**	**Bonds: Credit Spread; Options: Price; Loans: Probability of Default (PD) and Risk Weighted Assets (RWA)**
User Equilibrium; Link Flows ⇨			
	Random Process ⇨	⇨	
EXISTING TOOL			Δ Credit Spread; Δ Options Price; Δ PD and Δ RWA
ASSUMPTIONS:			
O-D Demand is deterministic.	Values for drift rate (trend) and volatility (variance) are assumed.	Sensitivity, i.e. elasticity between the capacity change and the link flow is linear.	Implementation of network improvement is certain (time wise), the size of the improvement is uncertain (see case study).

Figure 3-1: Network-Based Evaluation Method

The second step corresponds to the forecast of the future traffic and revenue on the tolled facility. The key assumption in this step is that the flow on the tolled links of interest follows a stochastic process, specifically geometric Brownian motion (GBM) (see section 4.2 for the properties and formulation of this random process). In this step, the uncertain traffic flow is correlated with the (uncertain) revenue by modeling the toll road revenue as a function of the traffic flow and a given toll rate. Consequently, since the traffic flow is uncertain and modeled as a random process, the toll road revenue can also be modeled as a random process, which allows the forecast of the uncertain future revenue stream.

Step three first deals with the identification of feeder and competing links via the calculation of correlations between the capacity on the tolled links and the flow on other network links, therefore allowing an assessment of the direction (i.e. positive or negative) of the impact of changes in the capacity of some links on the revenue risk. Then the actual effect of network improvement actions on the traffic flow or revenue on the toll road is quantified. Since the objective is to determine the sensitivity of the traffic flow or revenue on the tolled links to capacity changes on other links, the actual quantification is addressed via the calculation of the elasticity of the traffic flow or revenue on the tolled links to a change in the capacity on another link in the network.

The fourth step addresses the impact of changes in the network topology on the value of the toll road project by focusing on the impact that these changes would have on the traffic flow and the revenue on the tolled links. As expected, some changes in the network structure will positively affect the traffic flow on the tolled links thus increasing the revenue and reducing the price of the revenue risk, while others will detract from the value of the project thus increasing the price of the revenue risk. The difference between the estimated revenue risk under the base case (i.e. original network structure) and the estimated revenue risk after the occurrence of an improvement action to the network, allows the public sector to assess the impact of its network improvement decisions on the price of the revenue risk. This quantitative assessment also provides information to other participants in capital markets (i.e. private sector and financial institutions), which they can use in their risk assessment.

4. EVALUATION PROCESS

This chapter provides the detailed mathematical formulations of all steps in the developed decision support process. The implementation of these models is explained in detail in the context of two case studies, which are provided in the Appendix (i.e. Sioux Falls and Austin network).

4.1 Step 1: Traffic Assignment

As mentioned before, during the traffic assignment step, O-D demand is assigned to the network and the resulting traffic flows on the network links are obtained. Below, we present a background on traffic assignment under static conditions, highlight the challenges faced when using information on route flows, and discuss the concept of most likely route flows.

Consider a transportation network consisting of a set of nodes N and a set of directed links A (with $|A| = L$). Let A_J denote the subset of J links ($J \leq L$) where a toll p_j ($j = 1,.., J$) is charged. Define the generalized cost of travel (in time units) for those links subject to toll charges ($l \in A_J$) as follows[2]:

$$c_l(v_l, p_l) = t_l(v_l) + p_l / vot \tag{4-1}$$

where vot is the user's value of time (\$/min). Equation (4-1) shows that, in the case of the links subject to tolls, the generalized cost of travel includes both the travel time and a toll. In the case of the links not being charged ($l \in A, l \notin A_J$), the generalized cost of travel is given by the travel time.

Consider the case of fixed demand given by an Origin-Destination (O-D) matrix. Assume the link travel time functions $t_l(v_l)$ to be continuous and strictly convex on the link volumes, v_l. Further, assume both that the users are identical and value their time identically, and have perfect knowledge about their travel cost. Finally, let the users aim at minimizing their generalized cost

---

[2] Note that the definition of generalized cost (4-1), can be easily extended to deal with distance-based pricing, by thinking of p_j as a toll per distance unit and by writing $c_l(v_l, p_l) = t_l(v_l) + p_l \times l_l / vot$, where l_l is the length of link l.

of travel by choosing the shortest routes. Then, following Beckmann *et al*. (1956), the **User Equilibrium (UE)** link flows can be obtained by solving the following nonlinear convex optimization problem:

$$\underset{\mathbf{v}}{Min} \quad \sum_{l \in A} \int_0^{v_l} c_l(z, p_l) dz \tag{4-2}$$

subject to

$$\sum_{r \in R_w} h_r^{\ w} = d_w, \quad \forall w \in W \tag{4-3}$$

$$h_r^{\ w} \geq 0, \quad \forall r \in R_w, \ \forall w \in W \tag{4-4}$$

$$v_l = \sum_{w \in W} \sum_{r \in R_w} h_r^{\ w} . d_{lr}^{\ w}, \ \forall l \in A \tag{4-5}$$

where **V** corresponds to the vector of all the link flows (v_l, $l \in A$); W corresponds to the set of O-D pairs; w is a sub-index of O-D pairs; R_w corresponds to the set of routes that connect the origin-destination pair w; $h_r^{\ w}$ is the flow on route $r \in R_w$ that connects the origin-destination pair w; d_w corresponds to the (fixed) demand between the O-D *pair* w; and $d_{lr}^{\ w}$ is 1 if route r that connects the O-D pair w includes link l, and is 0 otherwise. The objective function of the above program (4-2) corresponds to the sum of the integrals of the link performance (i.e. generalized cost) functions. Constraint (4-3) indicates that the demand for travel between a particular O-D pair must be satisfied by using the routes that connect the O-D pair. Equations (4-4) and (4-5) correspond to the non-negativity and flow conservation constraints, respectively.

It is appropriate to discuss some considerations regarding the above formulation. First, one must note that the existence of a solution to the UE problem is ensured because the cost functions are continuous and the feasible set (4-3)-(4-5) is compact (Florian and Hearn, 2001). Second, in UE the (generalized) cost of travel for a given O-D pair is equal on all used routes and no lower on unused routes. Consequently, network users do not have incentives to change their routes since they cannot reduce their travel cost by choosing another route. One must also note that, when stating the user's problem as presented above, it is assumed that the link cost functions are separable and additive, and that all the routes that connect an O-D pair are perfect substitutes. Finally, it is also important to stress that, while (under the aforementioned technical considerations regarding the link cost functions) the optimal link flows are uniquely determined

in the above traffic assignment problem, the equilibrium route flows are not unique in general (i.e. trips assigned to routes between O-D pairs can be swapped to other routes while maintaining the link flows unchanged.). This is because when viewing the objective function (4-2) in terms of link flows, the objective function is strictly convex, so the vector of optimal link flows $\mathbf{v}^*$ that results from solving the optimization problem is unique. However, viewing the optimization problem in terms of route flow variables only, the objective function is only convex. Therefore, since route flows contain valuable information but are not unique, their analysis requires careful considerations (i.e. an additional behavioral assumption is needed in order to study route flows). In such a context, among all the UE route flow solutions, the entropy maximizing pattern is considered to represent the most likely one (Rossi *et al.*, 1989).

Most Likely Route Flows

Let R^* denote the set of all UE routes, and let R_w^* denote the set of UE routes that connect a given O-D pair w (i.e. $R^* = \bigcup_w R_w^*$). Then, for a given O-D pair w, the number of permutations giving rise to a set of route flows is given by:

$$\frac{d_w!}{\prod_{r \in R_w^*} h_r^w!} \tag{4-6}$$

Across all O-D pairs, the number of permutations is as follows:

$$E(h) = \prod_{w \in W} \frac{d_w!}{\prod_{r \in R_w^*} h_r^w!} \tag{4-7}$$

where $E(h)$ is known as the **entropy of the system**. Taking the logarithm of (4-7) and using Sterling's approximation, the following expression is obtained:

$$\ln E(h) = \sum_{w \in W} \ln d_w! - \sum_{w \in W} \sum_{r \in R_w^*} h_{r.}^w (\ln h_r^w - 1) \tag{4-8}$$

Finally, given the unique vector of UE link volumes $\mathbf{V}^*$, and recognizing that the O-D demand d_w is fixed, the optimization problem that must be solved for finding the route flow vector $\mathbf{h}^*$ that is most likely to occur (i.e. that maximizes the entropy of the system) can be expressed as follows:

$$\underset{h}{Max} \quad -\sum_{w \in W} \sum_{r \in R_w^*} h_r^w . \ln h_r^w \tag{4-9}$$

subject to

$$\sum_{r \in R_w^*} h_r^{\ w} = d_w, \quad \forall w \in W \qquad (4\text{-}10)$$

$$h_r^{\ w} \geq 0, \quad \forall r \in R_w^*, \ \forall w \in W \qquad (4\text{-}11)$$

$$v_l^* = \sum_{w \in W} \sum_{r \in R_w^*} h_r^{\ w} . d_{lr}^{\ w}, \forall l \in A \qquad (4\text{-}12)$$

Since the objective function (4-9) is concave, and the constraints are linear, the above program (4-9)-(4-12) has a unique solution, which is a global optimum of the problem (Bargera, 2006). Both primal and dual solution approaches for solving the above problem have been proposed in the literature. Regarding primal methods, such an approach is proposed by Bargera (2006). Among the dual methods, Janson (1993) presents a link-based procedure using successive Stochastic User Equilibrium approximations. Other dual methods that can be cited are the methods of iterative balancing (Bregman, 1957; Bell and Iida, 1997) and conjugate gradients (Larsson *et al.*, 2001). In this work, we employ Bregman's (1967) **iterative balancing method** for estimating the most likely path flows (see section 4.3 for details).

In the next step, given the traffic flows on the tolled links of interest obtained from traffic assignment, the random process used to model the traffic flow/demand and the revenue on the toll road is introduced.

4.2 Step 2: Forecast of Future Traffic Flow and Revenue on the Toll Road

The key assumption in this step is that the flow on the link of interest follows a stochastic process, in particular, geometric Brownian motion (GBM). This random process has been already used in the literature for this purpose (Brandao and Saraiva, 2008). We first present the definition of Brownian motion and then discuss the properties of GBM and its application in the context of this project.

Brownian motion (BM) is a stochastic process $\{X(t), t \geq 0\}$ with the following properties (Karlin and Taylor, 1975):

i.) Every increment $X(t+s) - X(t)$ is normally distributed with mean μt and variance $\sigma^2 t$; μ, σ being fixed,

ii.) For every pair of disjoint time intervals $[t_1, t_2], [t_3, t_4]$, where $t_1 < t_2 \leq t_3 < t_4$, the increments $X(t_4) - X(t_3)$ and $X(t_2) - X(t_1)$ are independent and normally distributed as mentioned above. Note that the same will apply if one considers n disjoint time intervals, where n is a positive integer.

iii.) $X(0) = 0$ and $X(t)$ is continuous at $t = 0$.

It is important to stress that, because every increment in the above stochastic process is normally distributed, which allows the underlying random variable to take negative values, it is not realistic to use BM for modeling the traffic demand/flow on the facility. Fortunately, GBM, which is defined as follows, resolves this issue:

$$S(t) = \exp[X(t)], t \geq 0 \tag{4-13}$$

The first two moments of GBM are:

$$E[S(t) \mid S(0) = s] = s \exp\left[t\left(\mu + \frac{\sigma^2}{2}\right)\right] \tag{4-14}$$

$$Var[S(t) \mid S(0) = s] = s^2 \exp\left[2t\left(\mu + \frac{1}{2}\sigma^2\right)\right]\left[\exp(t\sigma^2) - 1\right] \tag{4-15}$$

By assuming that the traffic flow follows geometric Brownian motion, one can then model the traffic demand on the facility as an uncertain process that evolves stochastically over time. From Equations (4-14) and (4-15), at a given point in time, the uncertain future traffic flow on the toll road can be modeled as lognormally distributed with known expected value and variance. In addition, by modeling traffic demand on the toll road as GBM, it is implicitly assumed that: (1) the expected traffic on the facility increases over time at some constant rate, and that (2) traffic demand on the facility at time $t+s$ depends only on the traffic observed at time t regardless of previous states.

Following Yescombe's (2002) discussion, if the traffic on the facility grows above an expected projection, it is probably not because of the project itself, but a result of the overall growth in the economy. Similarly, if the traffic is below a given projection, it can be attributed to an economic downturn. Hence, the realization of the traffic on the toll road might vary over time due to external factors, which are out of the control of project participants. Thus, it is realistic to assume that the realization of the traffic flow in the next time frame will depend on the current realization only.

Under the GBM assumption, the traffic demand/flow on the link of interest obeys the following stochastic differential equation:

$$dV_a = \mu V_a dt + \sigma V_a dW_t \qquad (4\text{-}16)$$

where V_a is the starting value of the traffic flow on the tolled link obtained from traffic assignment, μ is a drift rate, σ^2 is a variance, $dW_t = \sqrt{dt}\,\varepsilon_t$ is a Weiner process where dt is a time increment, and $\varepsilon_t \sim N(0,1)$. Consequently, the future traffic flow on the tolled link can be fully defined by knowing its starting value V_a, expected growth rate μ, and volatility σ. Irwin (2003) suggested that the values for the growth rate and the volatility can be extracted from past data, if such are available, or from similar projects. Here, it is assumed that these two values for the traffic flow process can be derived from similar projects and that they are constant over the concession period.

Stochastic Revenue Process

If the revenue is defined as a function of the traffic flow and the toll rate, it can be stated that the yearly revenue is equal to the yearly traffic on the tolled link times the toll rate. Then,

assuming both that V_a is expressed in annual traffic terms and the toll rate is a constant, the revenue function R can be defined as follows (Brandao and Saraiva, 2008):

$$R = V_a * T_r \qquad (4\text{-}17)$$

From Equations (4-16) and (4-17), and based on Ito's lemma, which is as follows (Trigeorgis, 1998):

$$dR = \frac{\partial R}{\partial t} dt + \frac{\partial R}{\partial V_a} dV_a + \frac{1}{2} \frac{\partial^2 R}{\partial V_a^2} \left(\sigma^2 V_a^2 dt \right) \qquad (4\text{-}18)$$

the revenue process can then be derived as a process which evolves stochastically over time as:

$$dR = \mu V_a T_r dt + \sigma V_a T_r dW_t \qquad (4\text{-}19)$$

The parameters that determine the behavior of the revenue over time are the starting annual traffic flow on the tolled link V_a, the expected traffic growth rate μ, the volatility σ, and the toll rate T_r.

4.3 Step 3: Impact of Improvement Actions on Competing and Feeder Links

This section first motivates and describes the proposed methodology for identifying competing and feeder links and routes. It also describes how elasticities are used to quantify the actual impact of changes in the network structure on the traffic flow and/or revenue on the toll road.

Consider the simple network shown in Figure 4-1, where the demand for trips between the O-D pair A-B is $d_{AB} = 4,000$, and the free flow travel times and link capacities for the links 1 and 2 are $t_1^0 = 15$ min and $C_1 = 1,000$, and $t_2^0 = 20$ min, and $C_2 = 2,000$, respectively. As shown in equation (4-20), BPR link performance functions with parameters $\alpha = 0.15$ and $\beta = 4$ are considered.

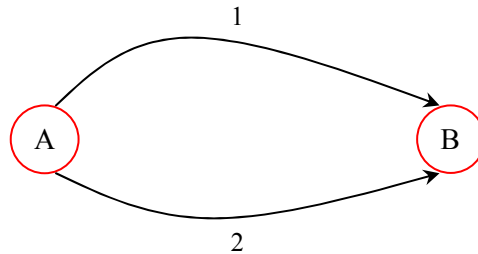$$t_a(v_a) = t_a^0 \cdot \left(1 + \alpha\left(\frac{v_a}{C_a}\right)^\beta\right) \tag{4-20}$$

Figure 4-1: Network 0

The equilibrium condition is given by Wardrop's (1952) first principle. Then, in User Equilibrium, the travel times on the two routes must be equal and demand must be satisfied:

$$t_1(v_1) = t_2(v_2) \tag{4-21}$$

$$v_1 + v_2 = d_{AB} = 4,000 \tag{4-22}$$

Solving these two equations, the equilibrium flows are found to be $v_1 = 1,521.909$ and $v_2 = 2,478.091$ vehicles/time unit. This solution can be depicted graphically as shown in Figure 4-2, where the link volumes are represented on the horizontal axis and the link travel times are represented on the vertical axis. The point of intersection of the link performance functions determines the equilibrium (link/route) flows, which correspond to the volumes that minimize the sum of the areas under the link performance functions.
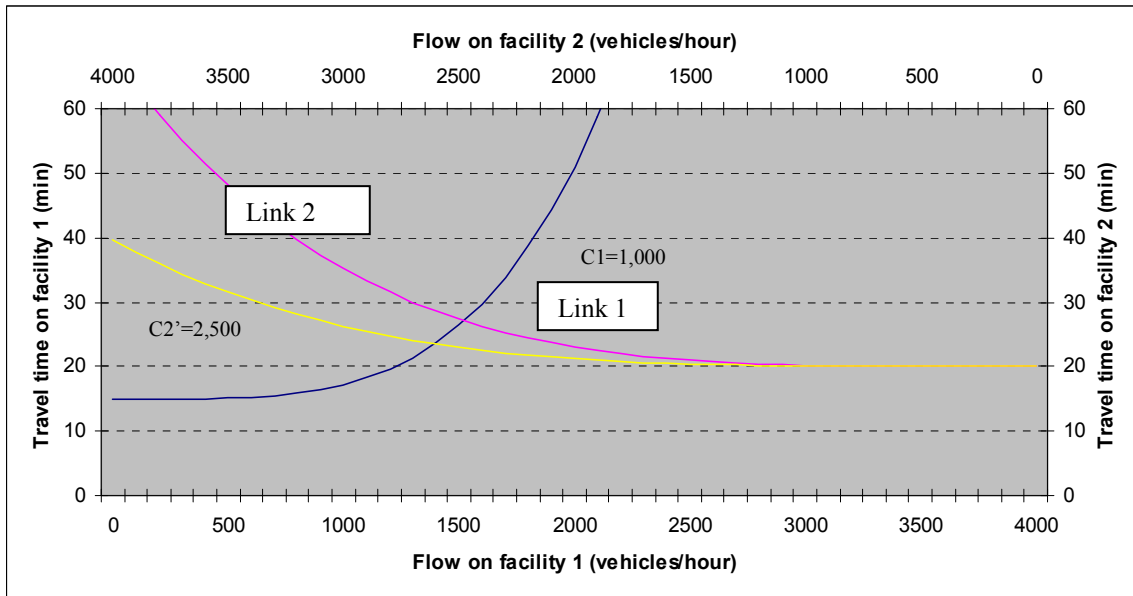
Figure 4-2: UE for Network 0

Now consider a new situation where the capacity on link 2 is assumed to be $C_2^1 = 2,500$ instead of $C_2^0 = 2,000$. As expected, since the BPR link travel time functions are decreasing functions of the link capacity, as the link capacity increases, less travel time will be needed to traverse link 2 at any traffic volume (greater than zero). Then, as shown in Figure 4-2, at the new point of equilibrium, the second facility attracts more traffic than in the original situation, while the traffic volume on the first facility is lower than in the original case. In fact, the new traffic volumes in equilibrium are $v_1 = 1,395.532$ and $v_2 = 2,604.468$ vehicles/time unit.

Now let us generalize the above observation to the context of a larger network. Intuitively, since the travel time on a route corresponds to the sum of travel times on all the links that constitute the route, as the capacity of a link (or set of links on the route) increases, the travel

time (and so the generalized travel cost) on the route decreases, therefore attracting traffic from **competing** routes and links. This will occur until a new point of equilibrium is reached, where the generalized travel cost among all the used routes connecting the O-D pair under consideration is equal again. One must also note that the traffic attracted from competing routes will reach the facility being considered via **feeder** routes and links.

The basic observation just presented, suggests a simple procedure for identifying competing and feeder routes/links by simulating variations in the capacity of the link (or set of links) of interest and studying the resulting correlation patterns. **Negative/positive** correlations between the capacity of the link (or set of links) of interest and the traffic volumes on other links are useful to **identify competing/feeder links**. **Negative/positive** correlations between the capacity of the link (or set of links) of interest and the most likely route flows on the routes connecting a given O-D pair are useful for **identifying competing/feeder routes**.

It is appropriate to stress that the use of simulated variations in capacity (at a given demand level) is just a tool that allows us to identify competing and feeder routes and links. In our approach, link capacity is not a source of uncertainty, it is exogenously given. As a consequence, since demand is the main source of uncertainty when planning toll roads, different demand level scenarios should be considered in the analysis. By comparing the results from different scenarios, the set of competing and feeder routes and links can be determined as described in later in this chapter. For the sake of completeness, before presenting the proposed methodology, we briefly discuss the concept of correlation as well as procedures for sampling from the normal distribution.

Covariance and Correlation

The covariance and correlation allow one to measure the association between two random variables. They provide information about the tendency of two variables to vary together rather than independently. Consider two random variables X and Y with corresponding means $E(X) = \mu_X$ and $E(Y) = \mu_Y$, and variances $Var(X) = \sigma_X^2$ and $Var(Y) = \sigma_Y^2$. The **covariance** of X and Y is defined as follows:

$$Cov(X,Y) = E[(X - \mu_x).(Y - \mu_Y)] \tag{4-23}$$

It can be shown that, if $\sigma_X^2 < \infty$ and $\sigma_Y^2 < \infty$, the above expectation will exist and $Cov(X,Y)$ will be finite. $Cov(X,Y)$ can be positive, negative or zero. If large values of X tend

to be observed with large values of Y, and small values of X tend to be observed with small values of Y, then $Cov(X,Y)$ will be positive. If large values of X tend to be observed with small values of Y, and small values of X tend to be observed with large values of Y, then $Cov(X,Y)$ will be negative.

Further, if $0<\sigma_X^2<\infty$ and $0<\sigma_Y^2<\infty$, the **correlation** (or **correlation coefficient**) of X and Y is defined as follows:

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X.\sigma_Y} \tag{4-24}$$

By using Schwarz inequality, it is possible to show that $-1 \le \rho(X,Y) \le 1$. It is also important to note that if X and Y are independent random variables with $0<\sigma_X^2<\infty$ and $0<\sigma_Y^2<\infty$, then $Cov(X,Y) = \rho(X,Y) = 0$. We must note that the converse is not true in general (i.e. we can have two dependent but uncorrelated random variables).

The value of $\rho(X,Y)$ provides a measure of the extent to which two random variables X and Y are linearly related. Given a series of n observations of the variables X and Y, the **sample correlation coefficient** can be calculated as follows:

$$\rho(X,Y) = \frac{\sum_{i=1}^{n}(x_i-\bar{x}).(y_i-\bar{y})}{(n-1)S_X.S_Y} \tag{4-25}$$

where S_X and S_Y are the sample standard deviations of X and Y.

Generating Samples from the Normal Distribution

As described in the next section, the methodology proposed here employs the generation of random variables and vectors to simulate variations in the levels of link capacities and demand. For simplicity, the normal distribution is employed in this research.

Generating Normal Random Variables and Vectors

Given a standard normal random variable Z (which we denote as $Z \sim N(0,1)$), a normal random variable $X \sim N(\mu,\sigma^2)$ can be obtained via the transformation $X = \mu + \sigma Z$. Therefore, we can focus on generating $N(0,1)$ random variables. Different methods can be used to generate

standard normal random variables, including the Box-Muller method (which is described below), the Polar method, and many others (see, for instance, Dagpunar, 1988).

The Box-Muller algorithm uses two independent and identically distributed (IID) random variables which are uniform in the interval (0,1) to produce two IID $N(0,1)$ random variables. The steps of the algorithm can be summarized as follows (Ross, 1997):

Step 1: Generate two IID $U(0,1)$ random variables U_1 and U_2

Step 2: Set $X = \sqrt{2\ln(U_1)}.\cos(2\pi.U_2)$ and $Y = \sqrt{2\ln(U_1)}.\sin(2\pi.U_2)$

In some cases, such as when dealing with different demand levels (i.e. different O-D matrices), we will be interested in generating a p-dimensional random vector $\mathbf{X}=(X_1, X_2,..., X_p)^T$ from a specified multivariate Normal distribution, where the individual components of the vector might not be independent. The p-dimensional multivariate normal distribution with mean vector $\mathbf{\mu} = (\mu_1, \mu_2,...., \mu_p)^T$ and Variance-Covariance matrix $\mathbf{\Sigma}$, whose (i, j)th entry is $\sigma_{ij} = \sigma_{ji} = Cov(X_i, X_j)$, has the following joint density function :

$$f(\mathbf{X}) = (2\pi)^{-n/2}|\mathbf{\Sigma}|^{-1/2}.\exp(-\frac{(\mathbf{X}-\mathbf{\mu})^T\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{\mu})}{2}) \qquad (4\text{-}26)$$

It is important to note that the Variance-Covariance matrix $\mathbf{\Sigma}$ is symmetric (i.e. $\mathbf{\Sigma^T} = \mathbf{\Sigma}$), positive semi-definite (i.e. $\mathbf{w^T}\mathbf{\Sigma}\mathbf{w} \geq 0$ for all $\mathbf{w} \in R^p$), and that its diagonal elements are non-negative (i.e. $\sigma_{ii} \geq 0$) (Law, 2007). The following algorithm can be employed to generate the desired multivariate normal vector $\mathbf{X}=(X_1, X_2,..., X_p)^T$:

Step 1: Generate $Z_1, Z_2,..., Z_p$ IID $N(0,1)$ random variables using the Box-Muller algorithm. These random variables can be expressed in vector notation as $\mathbf{Z}= (Z_1, Z_2,..., Z_p)^T$.

Step 2: Compute $\mathbf{X}=\mathbf{\mu}+\mathbf{C}\mathbf{Z}$, where $\mathbf{C}$ is a $p \times p$ lower triangular matrix such that $\mathbf{\Sigma} = \mathbf{C}\mathbf{C^T}$.

The unique factorization $\Sigma = CC^T$ (i.e. **Cholesky decomposition** of Σ) is possible since, as noted above, Σ is symmetric and positive semi-definite. The elements of **C** are obtained recursively as follows (Dagpunar, 1988);

$$c_{ii} = \sqrt{(\sigma_{ii} - \sum_{m=1}^{i-1} c_{im}^2)} \qquad (4\text{-}27)$$

$$c_{ji} = (\sigma_{ij} - \sum_{m=1}^{i-1} c_{im}.c_{jm})/c_{ii} \qquad (j > i) \qquad (4\text{-}28)$$

Based on the motivation presented at the beginning of this section, the following section presents the proposed methodology for identifying competing and feeder links and routes.

Identification of Competing and Feeder Links and Routes

Inputs:

o Transportation Network and its characteristics (i.e. link travel time functions, link capacities, free flow travel times, toll levels)

o (Original) O-D trip table

o Link of interest, to which we refer as link i (tolled link)

o iid sample of size N for the capacity on the link i (cap_i) under consideration. For simplicity, in this research, this iid sample is obtained by assuming that cap_i is normally distributed with a mean equal to the exogenously given capacity value, and assuming a certain standard deviation.

o In addition to the above inputs, if **variation around a given level of demand** is to be considered, a sample of W O-D matrices must also be generated. For simplicity, given the original O-D table, this is achieved by assuming that the demand for trips between O-D pairs follows a multivariate normal distribution with a given mean vector and variance-covariance matrix. Here, the mean vector is assumed to correspond to the values given in the original O-D table. For simplicity, the covariance matrix is specified by assuming that the variances are equal to the corresponding position in the mean vector after applying scaling factors and squaring them, and that the covariances are zero.

Outputs:

- o Sets of competing and feeder routes/links

The methodology proposed below is based on the study of the correlations between the capacity on the link of interest i (cap_i) and the traffic volumes on other links k (v_k), as well as the correlations between cap_i and the most likely path flows (MLPF) on UE routes q ($mlpf\_q$). Note that, two variations of the methodology are possible. In the first case, one can consider a **single level of demand** and then focus on studying variations in cap_i and their impact on traffic volumes and most likely path flows. In the second, one can consider **variations around a given level of demand** (i.e. around a given input O-D table) as well as variations in the capacity on the link of interest (cap_i). To a certain extend, the second approach is a very simple way to account for demand uncertainty when determining competing and feeder routes/links. Note that, the procedure shown below is written for the case in which different demand levels are considered, and that the procedure applicable for the single level of demand case is obtained by considering the original trip table (so that $W=1$) and by skipping Step 2 below.

Step 1: **Calculation of Correlation Tables**

For each of the w ($w=1,2,\dots,W$) realizations of demand (i.e. O-D matrices) do:

    For each of the j ($j=1, 2, \dots N$) realizations of the capacity on link i, $cap_i^{(j)}$, repeat:

    (i)   Run a traffic assignment (i.e. solve for UE) and identify the set of UE routes (i.e. identify those routes q that are used in equilibrium). Note that given the j-th realization of the capacity on link i, $cap_i^{(j)}$, one will obtain the vector of volumes on the L links of the network $\mathbf{V}^{*(j)} = [v_1^{(j)}, v_2^{(j)}, \dots, v_L^{(j)}]^T$. Further, note that at this point $v_l^{(j)} \geq 0$ ($l=1,\dots,L$).

    (ii)   Obtain/construct the vector of strictly positive link flows ($\mathbf{V}^{+*(j)}$) which will be used for estimating the most likely path flows (MLPFs). Note that the vector $\mathbf{V}^{+*(j)}$ contains only those links l such that $v_l^{(j)} > 0$, which are denoted as $v_l^{+(j)}$.

Also, note that $\mathbf{V}^{+^{*(j)}}$ is an $m_{(j)} \times 1$ vector (with $m_{(j)} \leq L$) where $m_{(j)}$ corresponds to the number of links of the network which are used in UE under $cap_i^{(j)}$.

(iii) Based on the set of UE routes, construct the link-path incidence matrix $\mathbf{\Delta}^{(j)}$ which corresponds to the system $\mathbf{V}^{+^{*(j)}} = \mathbf{\Delta}^{(j)}.\mathbf{h}^{(j)}$ (see equation 4-12). Note that $\mathbf{\Delta}^{(j)}$ is an $m_{(j)} \times n_{(j)}$ matrix where $n_{(j)}$ corresponds to the number of routes used in UE, and $\mathbf{h}^{(j)}$ is an $n_{(j)} \times 1$ vector of route flows. Also, based on the set of UE routes, construct the O-D demand-path incidence matrix $\mathbf{B}^{(j)}$ which corresponds to the system $\mathbf{T} = \mathbf{B}^{(j)}.\mathbf{h}^{(j)}$ (see equation 4-10). Note that $\mathbf{T}$ is a $p \times 1$ vector where p corresponds to the number of O-D pairs in the network, and $\mathbf{B}^{(j)}$ is a $p \times n_{(j)}$ matrix.

(iv) Determine the vector of MLPFs $\mathbf{h}^{*(j)}$ by using the iterative balancing algorithm presented at the end of this section.

(v) Determine the correlations between the capacity on the link of interest i and the volumes on other links k ($\rho_{capi,vk}$), and the correlations between the capacity on the link of interest i and the most likely route flows on routes q ($\rho_{capi,mlpf\_q}$).

Aside: Note that, at the end of Step 1, we will have W "correlation tables", so the w-th correlation table may look as shown below. Note that some positions of this table may be undefined.

$\rho^{(w)}_{capi,v1}$	$\rho^{(w)}_{capi,v2}$	$\rho^{(w)}_{capi,v3}$	…….	$\rho^{(w)}_{capi,mlpf\_1}$	$\rho^{(w)}_{capi,mlpf\_2}$	…….

Step 2: **Correlation Analysis** (Skip if W=1 and go to Step 3)

For each used path q that was identified over the W demand scenarios:

(i) Obtain $Max_{w=1,...,W} [(\rho^{(w)}_{capi,mlpf\_q})]$ and $Min_{w=1,...,W} [(\rho^{(w)}_{capi,mlpf\_q})]$ (i.e. the max and min values of the correlations between the capacity on the link i under consideration and the corresponding most likely route flow on route q).

(ii)   Compute the percentage of scenarios in which $\rho^{(w)}{}_{capi,mlpf\_q}$ was found to be: (1) undefined, (2) in the interval $[-1,-0.5]$, (3) in the interval $(-0.5,0.5)$, and (4) in the interval $[0.5,1]$.

(iii)   Calculate the following "weighted average" correlation index, $WACI\,(\rho_{capi,mlpf\_q})$

$$WACI(\rho_{capi,mlpf\_q}) = \frac{\sum_{w=1}^{W} g_w \cdot \rho^{(w)}{}_{capi,mlpf\_q}}{W} \qquad (4\text{-}29a)$$

$$\text{where } g_w = \begin{cases} 1 & \text{if } \rho^{(w)}{}_{capi,mlpf\_q} \text{ is defined in scenario } w \\ 0 & \text{otherwise} \end{cases}$$

Aside: Note that $WACI(\rho_{capi,mlpf\_q})$ is defined only when the correlation $\rho^{(w)}{}_{capi,mlpf\_q}$ exists in at least one of the demand scenarios. In such a case, $WACI(\rho_{capi,mlpf\_q})$ takes values in the interval [-1,1]. A highly negative value of the index (i.e. a value close to -1) is useful for identifying competing routes, while a highly positive value (i.e. a value close to 1) is useful for identifying feeder routes.

For each used link k that was identified over the W demand scenarios do:

(iv)   Obtain $Max_{w=1,...,W}\,[\,(\rho^{(w)}{}_{capi,vk})]$ and $Min_{w=1,...,W}[\,(\rho^{(w)}{}_{capi,vk})]$ (i.e. the max and min values of the correlation between the capacity on the link i under consideration and the link flow on link k).

(v)   Compute the percentage of scenarios in which $\rho^{(w)}{}_{capi,vk}$ was found to be: (1) undefined, (2) in the interval $[-1,-0.5]$, (3) in the interval $(-0.5,0.5)$, and (4) in the interval $[0.5,1]$.

(vi)   Calculate the following "weighted average" correlation index, $WACI\,(\rho_{capi,vk})$:

$$WACI(\rho_{capi,vk}) = \frac{\sum_{w=1}^{W} \gamma_w \cdot \rho^{(w)}{}_{capi,vk}}{W} \qquad (4\text{-}29b)$$

$$\text{where } \gamma_w = \begin{cases} 1 & \text{if } \rho^{(w)}{}_{capi,vk} \text{ is defined in scenario } w \\ 0 & \text{otherwise} \end{cases}$$

40

Note that a highly negative value of $WACI(\rho_{capi,vk})$ (i.e. a value close to -1) is useful for identifying competing links, while a highly positive value (i.e. a value close to 1) is useful for identifying feeder links.

Step 3: **Identification of Competing and Feeder Routes/Links**

By focusing on those O-D pairs connected by routes that use the link i under consideration, determine the percentage of demand scenarios in which the routes/links were identified as:

- Competing Routes: Routes q such that $-1 \leq \rho_{capi,mlpf\_q} \leq -0.5$ (i.e. those routes q that correspond to highly negative and significant correlations between the capacity of the link of interest i and the most likely route flow on the route).

- Feeder Routes: Routes q that: (1) use the link of interest i, and (2) such that $0.5 \leq \rho_{capi,mlpf\_q} \leq 1$ (i.e. routes q that correspond to highly positive and significant correlations between the capacity on the link of interest i and the most likely route flow on the route).

- Competing Links: Links k such that $-1 \leq \rho_{capi,vk} \leq -0.5$ (i.e. those for which the correlation between the capacity on the link of interest i and the traffic volume on link k is highly negative and significant).

- Feeder Links: Links k that: (1) belong to UE routes using the link of interest i, and (2) such that $0.5 \leq \rho_{capi,vk} \leq 1$ (i.e. those for which the correlation between the capacity on the link of interest i and the volume on link k is highly positive and significant).

Aside: One must note that, it is possible to have cases in which a route/link is regarded as a competing route/link at certain levels of demand (i.e. demand scenarios), and as a feeder route/link at other demand levels. We must also note that, in some cases, we might deal with "pure" competing routes/links and "pure" feeder routes/links.

Calculation of Most Likely Path Flows via an iterative balancing algorithm

In this section, the algorithm used to calculate the most likely path flows is described. The basis for the algorithm shown below is due to to Bregman (1967). Details about Bregman's algorithm can also be found in Lamond and Stewart (1981) and Jornsten and Lundgren (1989).

Inputs:

- o $\mathbf{V}^{+^*}$ and $\mathbf{\Delta}$, which are both constructed based on the outputs of UE by considering the links with strictly positive flow only. Recall that, $\mathbf{V}^{+^*}$ is an $m \times 1$ vector that includes only the volumes on links used in UE, and $\mathbf{\Delta}$ is an $m \times n$ link-route incidence matrix, where n is the number of UE routes.

- o $\mathbf{T}$ and $\mathbf{B}$, where T is a p x 1 vector of OD demands, where p is the number of OD pairs, and $\mathbf{B}$ is a p x n O-D demand-route incidence matrix, where n is the number of UE routes.

Outputs:

- o $\mathbf{h}^*$ (and the optimal dual vector $\mathbf{u}^*$). $\mathbf{h}^*$ is an $n \times 1$ most likely route flow vector (and $\mathbf{u}^*$ is an $(m+p) \times 1$ vector)

Note that the constraints (4-12) (flow conservation) and (4-10) (demand satisfaction), can be written, respectively, as:

$$\mathbf{V}^{+^*} = \mathbf{\Delta}.\mathbf{h} \tag{4-30a}$$

and

$$\mathbf{T} = \mathbf{B}.\mathbf{h} \tag{4-30b}$$

One can then summarize the above systems of equations as follows:

$$\boldsymbol{\pi} = \boldsymbol{\psi}.\mathbf{h} \tag{4-30c}$$

where $\boldsymbol{\pi} = \begin{pmatrix} \mathbf{V}^{+^*} \\ \mathbf{T} \end{pmatrix}$ is an $(m+p)$ x 1 vector, and $\boldsymbol{\psi} = \begin{pmatrix} \mathbf{\Delta} \\ \mathbf{B} \end{pmatrix}$ is an $(m+p)$ x n matrix.

The following notation is used below: ψ_j denotes the j-th row of the augmented matrix ψ; π_j denotes the j-th row ($j=1,2,...m, m+1, ...,m+p$) of the (augmented) vector π; u_j is the dual variable associated with the j-th constraint of system (4-30c); k is an iteration counter; and i

refers to the i-th constraint $\pi_i = \psi_i.\mathbf{h}^{*k}$ ($i=1,2,\ldots m$, $m+1$, …,$m+p$) of the system (4-30c). In this algorithm, the current solution is tested against one constraint at a time, and the corresponding dual variable is updated, if necessary, to satisfy the constraint. By testing all constraints cyclically and performing adjustments to the corresponding dual variables, the procedure converges (Jornsten and Lundgren, 1989). The algorithm is as follows:

Step 1: Initialization

$\quad k \leftarrow 0$

$\quad \mathbf{u}^k \leftarrow \mathbf{0}$

$\quad \mathbf{h}^{*k} \leftarrow \exp(+\boldsymbol{\psi}^{\mathbf{T}}.\mathbf{u}^k)$

$\quad$ Calculate $\boldsymbol{\psi}.\mathbf{h}^{*k}$ (i.e., $\psi_j.\mathbf{h}^{*k}$ for all j($j=1,2,\ldots,m,m+1,..,m+p$))

$\quad i \leftarrow 1$

Step 2: General Step (Balance of constraint i)

$\quad$ While $\psi_j.\mathbf{h}^{*k} \neq \pi_j$ for all constraints j (i.e., until convergence)

$\quad$ Repeat

$\quad\quad \gamma = -\ln(\pi_i /(\psi_i.\mathbf{h}^{*k}))$

$\quad\quad$ For $j=1,2,\ldots m,m+1,\ldots,m+p$:

$$u_j^{k+1} \leftarrow u_j^k - \gamma \ \text{ if } \ j=i$$

$$u_j^{k+1} \leftarrow u_j^k \quad\quad \text{ if } \ j \neq i$$

$\quad \mathbf{h}^{*k+1} \leftarrow \exp(+\boldsymbol{\psi}^{\mathbf{T}}.\mathbf{u}^{k+1})$

$\quad i \leftarrow (i \bmod m + p) + 1$

$\quad k \leftarrow k + 1$

$\quad$ End While

Calculation of Elasticities

In this research, the sensitivity of the flow or revenue on a tolled link to a change in the capacity on another network link is assessed through the estimation of elasticities. Thus, the calculated elasticities represent the percent change in the revenue or flow on a tolled link due to a percent change (increase or decrease) in the capacity on another network link. Denote the set of

links that constitute a given toll road facility as I and (as before) the set of all network links as A. Then the elasticity ε_{V_i,C_j} of the traffic flow V_i on the tolled link i ($i \in I$) to a change in the capacity C_j on link j ($i \neq j, j \in A, j \notin I$) can be expressed as follows:

$$\varepsilon_{V_i,C_j} = \frac{\dfrac{\Delta V_i}{V_i^0}}{\dfrac{\Delta C_j}{C_j^0}} \tag{4-31a}$$

where ΔV_i is the change in traffic flow on the tolled link i, V_i^0 is the initial traffic volume on the tolled link i, ΔC_j is the change in the capacity on link j, and C_j^0 is the initial value of the capacity on link j. Similarly, the elasticity ε_{R_i,C_j} of the revenue R_i on the tolled link i to a change in the capacity on link j can be obtained as:

$$\varepsilon_{R_i,C_j} = \frac{\dfrac{\Delta R_i}{R_i^0}}{\dfrac{\Delta C_j}{C_j^0}} \tag{4-31b}$$

where ΔR_i is the change in revenue on the tolled link i and R_i^0 is the initial revenue on the tolled link i.

Also note that, if the toll road facility consists of S links, then the elasticity ε_{TR,C_j} of the total revenue TR on the toll road ($TR = R_1 + R_2 + ... + R_S$) to a change in the capacity on link j can be obtained as follows (See Sydsaeter, 2005):

$$\varepsilon_{TR,C_j} = \frac{R_1^0 * \varepsilon_{R_1,C_j} + R_2^0 * \varepsilon_{R_2,C_j} + ... + R_S^0 * \varepsilon_{R_S,C_j}}{R_1^0 + R_2^0 + ... R_S^0} \tag{4-31c}$$

4.4 Step 4: Assessment of the Effect on Risk Dependent Value Measures

As discussed earlier, in this research, the effect of changes in the network structure on the value of the toll road project is analyzed through the assessment of the change in the price of the revenue risk. Further, the three financial instruments most commonly used in toll road agreements were chosen to illustrate the effect of the above mentioned changes. The following sections present detailed descriptions of the mathematical models used for pricing risk in the context of bank loans, bonds, and options.

Bank Loans

Bank loans are widely used in toll road agreements as the financial instrument for raising debt. These loans are suitable for the negotiation phase, when the project's revenues can only be estimated and are uncertain, because when the project has difficulties in the early stages of its operation phase, banks are willing to renegotiate the terms of the loan. The quality and the strength of the project to service its debt is estimated from different parameters, including the main risk– the revenue risk. The approach and the methodology that banks use for assessing risk in project finance agreements are addressed in this section. In addition, the identified key risk parameters are correlated with the stochastic approach used here to model the uncertain revenue stream.

Key Risk Parameters

As previously mentioned, the new Basel Accord (Basel Committee on Banking Supervision, 2005) recognizes the probability of default (PD), exposure at default (EAD) and loss given default (LGD) as key risk parameters in credit risk analysis. When calculating a loan's risk weight, in addition to considering these three risk parameters, banks need to forecast the level of potential credit losses. For this purpose, the new Basel Accord defines the distribution of real losses as the necessary input for the assessment of expected losses (EL) and unexpected losses (UL). EL represent the mean of the assessed distribution of losses and different banks'

provisions cover them. However, banks also need to cover a certain percentile of the assessed real losses (99.9[th] percentile). The difference between this threshold value and EL is considered as UL. It is assumed that, in times when the economy experiences a downturn, banks will be covered against 99.9% of all potential losses. In other words, banks will have a 0.1% probability of becoming insolvent.

Probability of Default

As noted before, it is assumed that default occurs when the borrower defaults on his debt obligations and stays below the debt level until the end of the loan's life. In other words, the project is considered to be in default when the revenue generated from the project drops below the debt repayment level and its projected path remains under the debt obligation's level. Denoting the loan's life as T, the modeling framework developed here defines the probability of default based on the event that the toll road revenue process (which follows GBM) hits a barrier D_l, where the barrier is below a starting revenue value $D_l < R(0) = r$. The probability that the revenue process hits the barrier D_l during the time interval $(0,T)$ and stays below D_l until the end of the concession life T (i.e. project defaults) can be expressed as:

$$P\{R(t) \leq D_l \cap R(T) \leq D_l\}, 0 \leq t \leq T \tag{4-32}$$

Real Losses

One must note that, a default event does not imply complete losses. In fact, exposure at default (EAD) and real losses must be assessed to determine the extent of the losses. Exposure at default (EAD) represents the amount of the loan at risk if default occurs. Again, since the project has a significant "residual value", this does not necessary imply that the whole amount will be lost. This is illustrated in Figure 4-3. The bank has invested some amount that is scheduled for repayment at some constant rate D_l and the project's revenue has dropped below the debt service level D_l in year t_l. Given that, until the default point the lender (bank) has received t_l repayments totaling $D_l t_l$, EAD is then:

$$EAD = D_l\left(T_l - t_l\right) \tag{4-33}$$
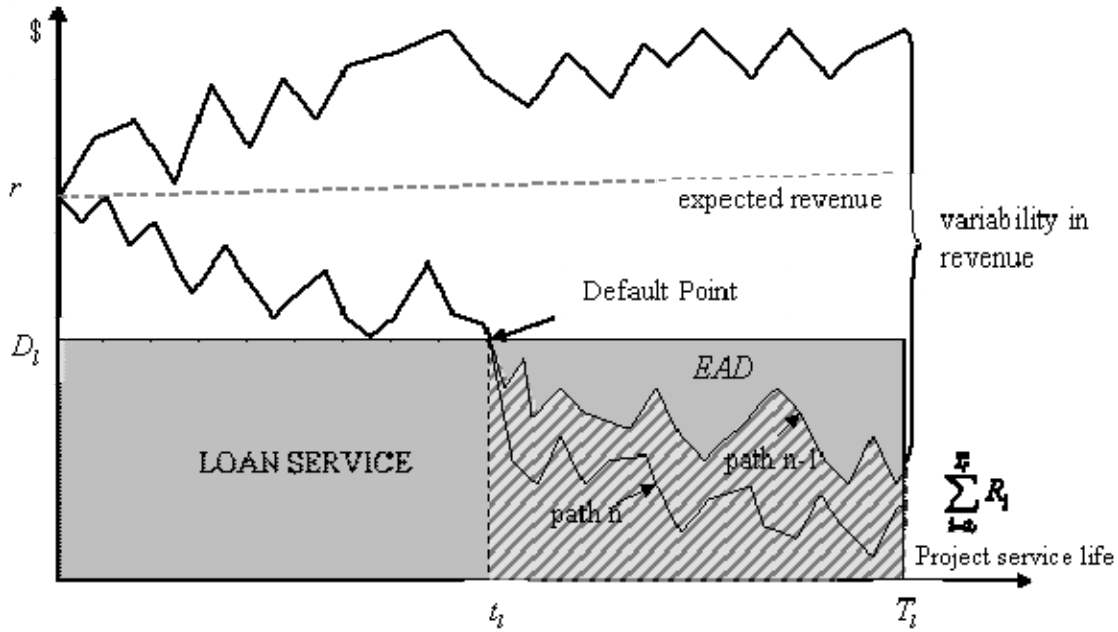
Figure 4-3: Default Point for Loans

Loss given default (LGD) is the ratio between the real losses and EAD. It is important to note that, when the default occurs, the bank is exposed at the level of EAD. However, the bank recovers some of the losses as the project continues to generate revenue over the remaining time horizon $(T_l - t_l)$. As the sum of the remaining revenue due to the continuation of the project's operation constitutes the amount that banks can recover, the real losses (RL) are then the difference between EAD and the amount that can be recovered:

$$RL = EAD - \sum_{i=t_l}^{T_l} R_i \tag{4-34}$$

From this equation, LGD can be expressed as:

$$LGD = \frac{RL}{EAD} = 1 - \frac{\sum_{i=t_l}^{T_l} R_i}{EAD} \tag{4-35}$$

Nevertheless, since the revenue behaves stochastically over time, the remaining revenue is also uncertain. Consequently, if the project defaults, it is uncertain how much the bank will be able to recover from the continuation of the project's operation. By using Monte Carlo simulation, if the number of simulated paths for the annual revenue over the project's service life is n and the number of simulated paths below the debt service level is m, the distribution of real losses can be determined for the sample of m paths as the difference between EAD and the sum of revenues for those paths (Equation 4-34). The mean of this distribution is EL and the 99.9^{th} percentile is the threshold level or Value at Risk (VaR).

Risk Weighted Assets

The New accord, Basel II, distinguishes two components of potential losses given default: Expected Losses (EL) and Unexpected Losses (UL) (Basel Committee on Banking Supervision, 2005). EL represent the average level of potential losses, and since they correspond to those losses that banks expect to occur, they are covered with different provisions. In order to derive the corresponding asset weight, Basel II recommends an upper level 99.9% confidence interval for the losses to be covered by capital reserves. The value of the loss expected to be exceeded with 0.1% probability is known as the Value-at-Risk (VaR) (Linsmeier and Pearson, 2000).

Basel II requires banks to provide capital requirements that will buffer UL at the VaR level. Capital requirements are set to be minimum 8% of UL (Basel Committee on Banking Supervision, 2004).

$$CAR = \frac{\text{Regulatory Capital (Tier I)}}{RWA} \tag{4-36}$$

where CAR is the Capital Adequacy Ratio (equal to 8%), and RWA are Risk Weighted Assets (Basel Committee on Banking Supervision, 2005).

In order to calculate RWA, and without loss of generality, it is assumed that: 1) the loan is not correlated with any systematic risk factor, and that 2) maturity adjustments are not applicable. Note that, Basel II takes into account the correlation of the asset with systematic risk factors when calculating the capital requirements (Basel Committee on Banking Supervision, 2005). However, in this research's context, the revenue is the underlying variable for the credit

risk assessment of the project finance loan, and since the revenue from toll roads cannot be considered as an asset nor is it traded on the market (Irwin, 2003), the correlation between the loan (which depends on the estimated project revenue) and the systematic risk factors is not considered here. Regarding the second assumption, one must note that the maturity adjustment in the calculation of the capital requirements addresses the issue of potential downgrades over long-term credits, where these downgrades represent the probability that the borrowers will move from one rating grade to another, or, in other words, that the probability of default will change over a given time horizon. However, since the assessment of credit risk in revenue-generating projects is dependent on the revenue, and the uncertainty in these cash flows is already being taken into account by modeling it as a stochastic process (therefore also capturing the uncertainty that affects the probability of default over time), the maturity adjustments are not considered here. Note that, since credit worthiness is related to the forecasted cash flows only, the financial viability of the project is fully captured by the traffic and revenue models presented before.

UL are the difference between EL and VaR (Basel Committee on Banking Supervision, 2005), where EL is the expected value of the distribution of real losses given default, and VaR is the 99.9^{th} percentile of the same distribution:

$$UL = VaR - EL \tag{4-37}$$

Here, it is important to note that the capital of the bank is set to cover the gap between EL and VaR (Basel Committee on Banking Supervision, 2005). In order to calculate RWA, this gap needs to be multiplied by a factor of 12.5 (i.e. reciprocal of the minimum capital requirements ratio which equals 8%):

$$RWA = 12.5 * PD * UL \tag{4-38}$$

where PD is the probability that the project will default in a given year and UL are derived as described above. Here, one must stress that UL are derived from the distribution of real looses given that the project has defaulted in a given year (i.e. the distribution of real losses is conditional on default).

Bonds

As discussed earlier, there are several approaches to model and price credit risk. The common underlying goal in all these models is to forecast the probability of default based on some assumptions.

The credit migration approach is based on the assessment of the probability that the borrower will move from one credit grade to another (including default) over a period of time, where the assessment is made by considering the value of the borrower's portfolio (Crouhy *et al*., 2000). The initial step is to assign a rating grade to the borrower and then estimate the probability of moving from one grade to another. The initial rating grade can be accepted based on a rating by agencies such as Moody's, Standard & Poor and Fitch.

The actuarial approach is based on modeling default as a Poisson process. In this approach there are no assumptions on why the borrower defaulted, and the capital of the firm is not related to the default risk. The input to this model is the mean of a Poisson distribution or the expected number of defaults within a given time horizon.

The CreditPortfolioView model is based on the estimation of macroeconomic variables which are specific to each country. The probability of default is conditional on these macroeconomic variables and modeled as a logit function.

The structural approach is based on the asset value or the market value of the firm, assuming that the value follows geometric Brownian motion and equals the sum of future cash flows (Giesecke, 2004). The probability of default occurs when the value drops below the debt level. Here, two situations can be distinguished. First, when the value drops below the debt level and remains below it for the debt service life, it is considered that the obligor has defaulted. This approach to the definition of default is called classical approach. On the other hand, the first-passage approach defines default when the value drops below the debt level for the first time, thus not observing the remaining debt service and the behavior of the value over the remaining time period.

These four models were analyzed to examine their applicability to pricing credit risk in this research, and after identifying some of these models' limitations, the structural approach was adopted as the model for the assessment of the credit risk in the context of toll road projects. Note that, a toll road is a stand alone facility whose value is derived from its forecasted revenues.

By modeling revenue as geometric Brownian motion, the uncertainty in the future revenue outcome can be captured, and consequently, the probability of default can be assessed for the whole concession life. Thus, in our modeling context, assessing the probability of default based on the migration of the project's value from one credit grade to another or by assuming a Poison process are not applicable. Further, although different macroeconomic variables do in fact affect traffic demand, one can assume that the traffic studies, which are prepared before the concession agreement is offered, will take into account possible variations in the economic environment, therefore allowing the assessment of the expected traffic growth rate that is used in our proposed modeling framework. The key risk parameters of this model and its application in the context of this research are discussed in the following section.

Key Risk Parameters

The main principles in the structural approach for credit risk assessment date back to Merton (1974). In such context, the value of the firm is both a stochastic variable over time and the main driver of the credit risk. It is also assumed that an asset can be traded instantaneously and that the debt liability is a claim on the asset. The main risk parameter is the probability of default, or the probability that the obligor will default to repay the debt. Further, in order to price the credit risk, the price of the risky debt is compared to the price of the risk free debt. The difference between the rates of return on these two debts, risky and risk free, is the credit spread.

Application of this approach to pricing revenue risk in toll roads requires some assumptions. The first assumption is that, the revenue, modeled as a stochastic process and treated as the source for the debt repayment, is the claim for the debt liability. This means that, in case of default, the bond investor will receive the remaining revenue collected from the continuation of the toll road service. Second, the default occurs when the revenue process drops below the debt level and remains below it for the debt service life (classical approach). Since the definition of default is equal to the one previously discussed in the analysis of bank loans, the following section explains the model for pricing risk in detail, that is, the model for the assessment of the credit spread.

Credit Spread

The Investors in a bond market observe the credit spread of an offered bond as the parameter for assessing their return on investment and associated risk. The investor makes a decision about purchasing the bond based on the risk and adequate compensation as reflected in the price of the bond. In such context, the yield or rate of return r the investor will earn from his/her investment can be calculated from the following expression:

$$P = \sum_{t=1}^{T} \frac{C}{(1+r)^t} + \frac{K}{(1+r)^T}$$ (4-39)

where P is the selling price of the bond, K is the face value, C is a coupon payment, and T is the bond's maturity.

Credit spread is the difference between the yield of a risky bond and the yield of a risk free bond. In theory, it can be assumed that there exists a risk free bond and practitioners commonly use the yield of a government or a treasury bond with similar maturity for the calculation process. A risky bond can be defined as a bond that has some probability of default. As defined previously, the probability of default is the probability that the bond issuer will fail to repay a promised coupon rate or the principal at the maturity. The price of the bond is determined in such a way that the associated risk is included. In other words, the price of the risky bond will be lower than the price of the risk free bond because of the reduction caused by the price of the associated risk, and consequently, the yield of the risky bond will be higher than the yield of the risk free bond. The explanation of the pricing method for revenue-generating projects is presented in Figure 4-4.
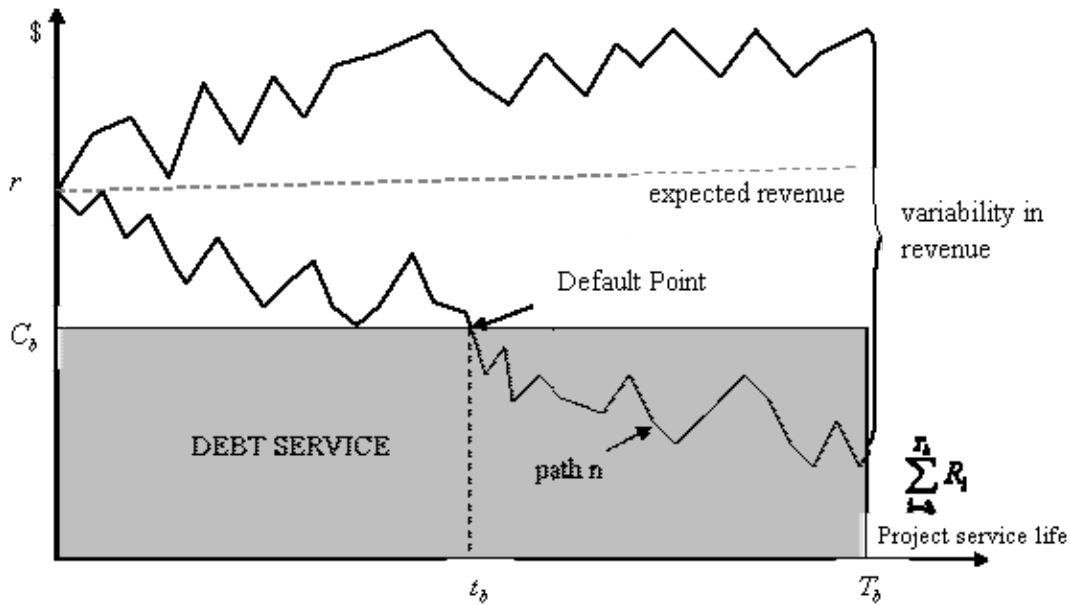
Figure 4-4: Default Point for Bonds

Define the total amount of debt as D_b, and assume both that the loan is planned to be repaid through equal yearly repayments C_b and that the debt life is T_b. In the developed methodology framework, the probability of default is defined based on the event that the revenue hits the barrier C_b (where this barrier is below the starting revenue value i.e. $C_b < R(0)$) and remains below the barrier. If the revenue R is higher than C_b for the whole debt service life T_b, then the bondholder receives all promised payments. If R drops below C_b at some time $t_b \in [0, T_b]$, the bondholder will receive the remaining revenue and experience a loss of:

$$C_b(T_b - t_b) - \sum_{i=t_b}^{T_b} R_i \qquad (4\text{-}40)$$

This position of the bondholder is the same as if he would have sold an American put option with strike price $C_b(T_b - t_b)$ and maturity T_b. An American option is a type of option that can be exercised at any time during the option's life. This is equivalent to the definition of default as it can occur at any time of the debt service. Further, a put option is a type of option that is used for risk hedging when the price of the asset is expected to decrease. The seller of the put option accepts the obligation to buy the underlying asset for the pre-specified price (i.e. the

exercise price) if the value of the asset drops below some level (i.e. strike price). If the asset price stays above the set barrier (strike price) during the bond tenure, the seller earns full provision for selling the put option.

Thus, the bond investor accepts the risk that the revenue will drop below the debt service level, and this risk is incorporated into the exercise price. The value of the option reflects the price of the default risk and is determined as the average over all possible realizations. The realization is positive for the bondholder when he receives the payment as promised and there is no risk, thus the value of the option is 0. If the realization is negative, the bondholder will receive only the available amount. Thus, the value of the option is equal to the difference between the strike price and the available amount.

In the context of toll roads and revenue risk, the price of the option to accept the revenue risk is equal to:

$$\max\left(0, C_b\left(T_b - t_b\right) - \sum_{i=t_b}^{T_b} R_i\right) \tag{4-41}$$

where, as discussed above, $C_b\left(T_b - t_b\right)$ is the remaining value of the debt repayment at time of default t_b, and $\sum_{i=t_b}^{T_b} R_i$ is the sum of the remaining revenue. The value of the risky bond P_r is then equal to the value of a risk free bond P_f reduced by the value of the put option:

$$P_r = P_f - \max\left(0, C_b\left(T_b - t_b\right) - \sum_{i=t_b}^{T_b} R_i\right) \tag{4-42}$$

Thus, the credit spread can be obtained as the difference between the risky rate r_r and the risk free rate r_f, where the rate of return r_r for the holder of the risky bond can be derived according to Equation (4-39). As discussed earlier, the focus of this research is on pricing the revenue risk and how the price of this risk is affected by the changes in the network topology. Hence, the credit spread is chosen as the risk measure of interest for bonds. The sensitivity of the credit spread to changes in the capacity on a subset of network links is addressed in this research.

Buy-Back Option

Real options are used in project finance as a tool for risk mitigation. From the literature review, it was noted that the form of the option used in toll road agreements varies from different government guarantees to revenue sharing agreements. Also, different mathematical approaches are used. This suggests that the private and the public sectors do not have a standard type of model to be used in toll road agreements, and that the models may vary from project to project. Hence, within the area of financial instruments, this research was focused on the development of a new approach for option value assessment. The option to buy-back the project is analyzed in detail in the following section.

Option Valuation

For the public sector, the option to buyback the project is a right to acquire the project back to its ownership if the profit from the project's operation exceeds some predetermined level. In this case, the owner (the public sector) has the right to buy-back the project for some value and continue to operate the project and collect all future revenues. Since one of the parameters for the valuation of options is the value of the underlying project (and given that the uncertainty associated with the revenue process also translates into uncertainty in the project's value), the value of the project is determined as the expected sum of the uncertain future revenues.

Here, the condition for exercising the option is determined by the event when the average revenue (AR) over a given time horizon exceeds a pre-specified upper boundary. This approach, which considers the average revenue in calculating the option's value, overcomes the problem of annual traffic volatility, hence revenue volatility risk too. It allows the traffic flow and revenue to fluctuate over a pre-specified period and takes into the consideration the average revenue value only.

For the period under consideration, AR is calculated by using Monte Carlo simulation as the sum of the discrete values of the annual revenue for each simulated path divided by the length of the time horizon. Once the value of AR is evaluated and set, the next step is to compare

the expected value of the project with the exercise price for those simulation paths for which the forecasted revenue is above AR.

The expected value of the integral of the GBM process is derived below. This formulation is used for the calculation of the expected value of the project over the remaining concession period. Consider a stochastic process in a risk neutral environment that obeys the following stochastic differential equation:

$$dS(t) = rS(t)dt + \sigma S(t)dW_t \qquad (4\text{-}43)$$

where r is the risk-free rate, and σ is the volatility. Define the time to the maturity of an option as T and assume that both r and σ are constant over the time horizon $[t_0, T]$. Now consider the integral of the stochastic process over the observed time horizon:

$$A(T) = \int_{t_0}^{T} S(y)dy \qquad (4\text{-}44)$$

With the definition of the expected value of the underlying process:

$$E\big[S(T)\big] = S_{t_0} * \exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right] \qquad (4\text{-}45)$$

Equation (4-44) becomes:

$$E\big[A(T)\big] = E\left[S_{t_0} \int_{0}^{T-t_0} \exp\left[\left(r - \frac{1}{2}\sigma^2\right)y + \sigma W_y\right]dy\right] \qquad (4\text{-}46)$$

Geman and Yor (1993) show that the first order moment of the process $A_t^{(v)} = \int_0^t \exp\big[2(vs + W_s)\big]ds$ is equal to:

$$E(A_t^{(v)}) = \frac{1}{2(v+1)}\big[\exp(2t(v+1)) - 1\big] \qquad (4\text{-}47)$$

where

$$v = \frac{2r}{\sigma^2} - 1 \qquad (4\text{-}48)$$

Due to the scaling property of Brownian motion $W(t) = (1/a)W(ta^2)$ and using the change of variable $y = 4v/\sigma^2$ (Geman and Yor, 1993), Equation (4-46) becomes:

$$E[A(T)] = \frac{4S_{t_0}}{\sigma^2} E\left[\int_0^{\frac{4}{\sigma^2}(T-t_0)} \exp(2(vy+W_y))dy \right]$$ (4-49)

Now consider $S_{t_0} = R_{t_0}$ and regard $A(T)$ as the value of the project, then the expected value of the project $E[PV]$ at time t_0 is:

$$E[PV(t_0)] = \frac{4R(t_0)}{\sigma^2} \frac{[\exp(2h(v+1))-1]}{2(v+1)}$$ (4-50)

where

$$h = \frac{\sigma^2}{4}(T-t_0)$$ (4-50a)

$$v = \frac{2\alpha}{\sigma^2} - 1$$ (4-50b)

Now let us denote an upper bound as UB and define a time interval $[0,t_0]$ in which the average revenue value is calculated. The option is priced as a European barrier option, which means that it can be exercised only at the end of the option's life (i.e. at time t_0). The option to buyback the project is considered as an up-and-in barrier option, which means that, if AR is above UB (up), the option can be exercised (in). For each simulated path, if AR in $[0,t_0]$ is above U, the value of the option is calculated by comparing the expected value of the project for the remaining time period with the exercise price:

buy-back option (European call option)

$$C = \left[\max\left(E[PV(t)]-K_c, 0 \right) \mid \overline{AR}^n(t_0) > UB \right]$$ (4-51)

where Kc is the exercise price and C is the value of the corresponding option. The exercise price can be regarded as the cost of the initial investment in the project (Garvin and Cheah, 2004).

The public sector can exercise the option for those paths where the buyback option is in the money (i.e. when the expected value of the project is higher than the exercise price). In such cases, the public sector expects that the project will generate more profit than the payment required for the option (exercise price). The option valuation model is presented in Figure 4-5.
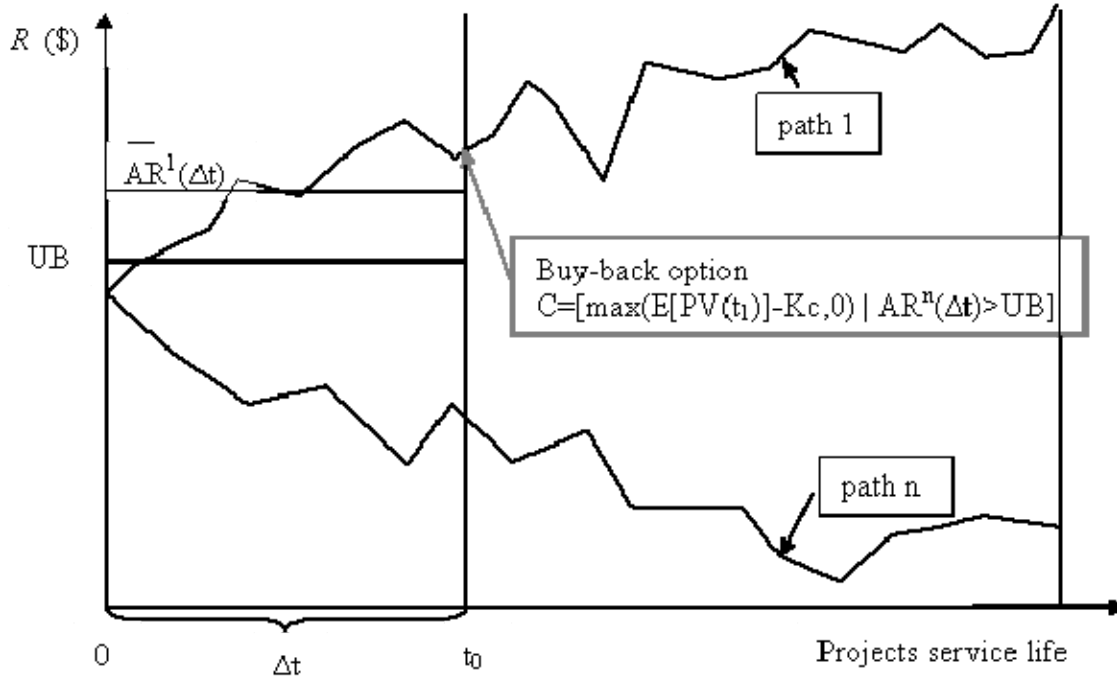
Figure 4-5: Pricing the Option to Buy Back

Among all the simulated revenue paths, let us focus our attention on a specific path, say path 1. After time Δt, when the option can be exercised, the value of $\overline{AR}^1(\Delta t)$ is determined as the average of all simulated revenue paths within the period $[0, t_0]$. This average revenue is then compared to UB, which is set in advance. Since $\overline{AR}^1(\Delta t) > UB$, the option becomes alive and can be exercised if it has a positive payoff. For the simulated path 1, the expected value of the project is determined from Equation (4-51) and then compared to the exercise price Kc.

5. CONCLUSIONS

Development, delivery, and operation of public infrastructure are becoming increasingly dependent on participation of the private sector. While revenue generating projects, such as toll roads, were traditionally developed and funded from the public sources, in recent years, as the public demand for new projects have exceeded the ability of the public sector to deliver them, the private investors have started to fulfill the gap between the needed and the available infrastructure.

However, the participation of the private sector in PPP agreements is constrained by the availability of funds, thus putting the financial institutions in the role of lenders in these agreements. The schemes of PPP agreements and the interaction among project participants are very complex. Due to this complexity, there are numerous associated risks, but one risk can be distinguished as the core of the project's financial feasibility: the revenue risk.

This research examines the correlation of the revenue risk and the three financial instruments most commonly used in PPP agreements: bank loans, bonds, and real options. A new methodology that links transportation network planning parameters with financial risk measures is proposed. This network-based valuation method provides a quantitative assessment of the impact of changes in the network structure on the financial feasibility of toll road projects.

The valuation method developed in this research is a valuable tool, which has the potential to add to current practices of toll road modeling due to the following:

- It creates a solid basis for the objective assessment of the impact that decisions made by the public sector, regarding network improvements, have on financial instruments;
- It provides a tool for quantification of the impact of changes in the network structure on the project's financial feasibility; and
- It builds a common ground which ensures that the decisions made by the public sector are interpreted in a meaningful manner by project managers and decision makers, that is, the public sector, the private sector, and financial institutions.

6. REFERENCES

Alexander, I., Estache, A., and Oliveri, A. (2000). "A few things transport regulators should know about risk and the cost of capital." *Utilities Policy*, 9(1), 1-13.

Arnott, R., De Palma, A. and Lindsey, R. (1990). "Economics of a bottleneck." *Journal of Urban Economics*, 27(1), 11-30.

Arnott, R., De Palma, A. and Lindsey, R. (1993). "A Structural model of peak-period congestion: A traffic bottleneck with elastic demand." *American Economic Review*, 83(1), 161-179.

Arnott, R., De Palma, A. and Lindsey, R. (1994). "The welfare effects of congestion tolls with heterogeneous commuters." *Journal of Transport Economics and Policy*, 28, 139-161.

Bar-Gera, H. (2006). "Primal Method for Determining the Most Likely Route Flows in Large Road Networks." *Transportation Science*, 40(3), 269-286.

Basel Committee on Banking Supervision (BCBS) (2004). "International Convergence of Capital Measurement and Capital Standards. A Revised Framework." *Bank for International Settlements*, Basel, Switzerland.

Basel Committee on Banking Supervision (BCBS) (2005). "An Explanatory Note on the Basel II IRB Risk Weight Functions." *Bank for International Settlements*, Basel, Switzerland.

Beckmann, M., McGuire, C. and Winsten, C. (1956). "Studies in the Economics of Transportation." *Yale University Press*, New Haven.

Bell, M. and Iida, Y. (1997). "Transportation Network Analysis." *John Wiley & Sons*, Chichester, England.

Bell, M.G.H and C. Cassir. (2000). Reliability of Transport Networks. Research Studies Press Limited.

Braid, R.(1989). "Uniform versus peak-load pricing of a bottleneck with elastic demand." *Journal of Urban Economics*, 26, 320-327.

Braid, R.(1996). "Peak-load pricing of a transportation route with an unpriced substitute." *Journal of Urban Economics*, 40, 179-197.

Brandao, L.E.T., and Saraiva, E. (2008). "The Option Value of Government Guarantees in Infrastructure Projects." *Construction Management and Economics,* 26(11), 1171-1180.

Brealey, A. R., Cooper, A.I. and Habib, A.M. (1996). "Using Project Finance to Fund Infrastructure Investments." *Journal of Applied Corporate Finance,* 9(3), 25-39.

Bregman, L. (1967). "The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming". *U.S.S.R. Computational Math. Mathematical Phys.*, 7, pp. 200-217.

Chen, A., Subprasom, K., and Chootinan, P. (2001). "Assessing financial feasibility of build-operate-transfer project under uncertain demand". *Transportation Research Record*, 1771, 124–131.

Chen, A., Subprasom, K., and Ji, Z.-W. (2003). "Mean-variance model for the build-operate-transfer scheme under demand uncertainty". *Transportation Research Record*, 1857, 93–101.

Chen, A., and Subprasom, K. (2007). "Analysis of regulation and policy of private toll roads in a in a build-operate-transfer scheme under demand uncertainty". *Transportation Research Part A*, doi:10.1016/j.tra.2006.11.009.

Chen, A., H. Yang, H. Lo and W.H.Tang. (1999). "A Capacity related reliability for transportation networks." *Journal of Advanced Transportation*, 33(2), 183-200.

Chiara, N., and Garvin, M.J. (2007). "Using Real Options for Revenue Risk Mitigation in Transportation Project Financing." *Transportation Research Record: Journal of the Transportation Research Board, No. 1993*, Transportation Research Board of the National Academies, Washington, D.C., 1-8.

Chow, J.Y.J, and Regan, A. (2009) "Real Option Pricing of Continuous Network Design Investment." Transportation Research Board Annual Meeting (CD-ROM), Washington D.C.

Clark, S., and Watling,D. (2005). "Modelling network travel time reliability under stochastic demand." *Transportation Research Part B*, 39, 119-140.

Crouhy, M., Galai, D., and Mark, R. (2000). "A comparative analysis of current credit risk models." *Journal of Banking & Finance,* 24, 59-117.

Dafermos, S.C., 1973. "Toll patterns for multiclass-user transportation networks." *Transportation Science*, 7, 211-223.

Dagpunar, J. (1988). "Principles of Random Variate Generation." *Oxford University Press*, Oxford, England.

De Palma A, and Lindsey, R. (2000). "Private toll roads: competition under various ownership regimes." *Annals of Regional Science*, 34, 13–35.

Dial R.B. (1971). "A Probabilistic multipath traffic assignment model which obviates path enumeration." *Transportation Research*, 5, 83-111.

Du, Z.P. and A. Nicholson. (1997). "Degradable Transportation Systems: Sensitivity and Reliability Analysis." *Transportation Research Part B*, 31, 225-237.

Duthie, J., A. Unnikrishnan, and S. T. Waller, 2007. "Network Evaluation with Uncertain and Correlated Long-Term Demand." In Review.

Edwards, P. and Bowen, P. (2003). "Risk perception and communication in public-private partnerships." *Public Private Partnerships: Managing risks and opportunities,* Akintoye, A., Beck, M. and Hardcastle, C., eds, Blackwell Science Ltd, Oxford, UK, 79-91.

Fishbein, G. and Babbar, S. (1996). "Private Financing of Toll Roads." *RMC Discussion Paper Series,* No 117, The World Bank.

Florian, M. and Hearn, D.W. (2001). "Traffic Assignment: equilibrium models". *Proceedings of the Institute of Mechanical Engineers*, 215(1), 305-315.

Flyvbjerg, B., Holm, M.K.S, and Buhl, S.L. (2005). "How (In)accurate Are Demand Forecasts in Public Works Projects?" *Journal of the American Planning Association,* 71(2), 131-146.

Gatti, S., Rigamonti, A., Saita, F., and Senati, M. (2007). "Measuring Value-at-Risk in Project Finance Transactions." *European Financial Management,* 13(1), 135-158.

Gardner, L., Unnikrishnan, A., and Waller, S.T. (2007) "Robust Pricing of Transportation Networks under Uncertain Demand." In Review.

Garvin, M. and Cheah, C.Y.J. (2004). "Valuation techniques for infrastructure investment decisions." *Construction Management and Economics*, 22(5), 373-383.

Geman, H., and Yor, M. (1993). "Bessel Processes, Asian Options and Perpetuities." *Mathematical Finance*, 3(4), 349-375.

Geman, H. and Yor, M. (1996). "Pricing and Hedging Double-Barrier Options: A Probabilistic Approach." *Mathematical Finance,* 6(4), 365-378.

Giesecke, K. (2004). "Credit Risk Modeling and Valuation: an Introduction." *Cornell University.*

Hardcastle, C. and Boothroyd, K. (2003). "Risks overview in public-private partnership." *Public Private Partnerships: Managing risks and opportunities,* Akintoye, A., Beck, M. and Hardcastle, C. (eds), Blackwell Science Ltd, Oxford, UK, 31-56.

Hearn, D.W., Ramana, M.V. (1998). "Solving congestion toll pricing models." *Equilibrium and Advanced Transportation Modeling*, P. Marcotte, S. Nguyen (eds.), Kluwer Academic Publishers, Boston, 109–124.

Hearn, D.W., Yildirim, M.B. (2001). "A toll pricing framework for traffic assignment problems with elastic demands." Current Trends in Transportation and Network Analysis: Miscellanea in Honor of Michael Florian, M. Gendreau, P. Marcotte (eds.), Kluwer Academic Publishers, Dordrecht, The Netherlands.

Irwin, T. (2003). "Public money for private infrastructure: Deciding when to offer guarantees, output-based subsidies, and other fiscal support." *World Bank working paper No 10,* Washington D.C., 41-51.

Janson, B. (1993). "Most Likely Origin-Destination Link Uses from Equilibrium Assignment." *Transportation Research Part B*, 27(5), 333-350.

Joksimovic, D., Bliemer, M., and Bovy, P. (2005). "Optimal Toll Design Problem in Dynamic Traffic Networks with Joint Route and Departure Time Choice". *Transportation Research Record*, 1923, 61-72.

Jornsten, K. and Lundgren, J. (1989). "An entropy-based modal split model". *Transportation Research Part B*, Vol. 23, No. 5, pp.345-359.

Karlin, S. and Taylor, H.M. (1975). "A First Course in Stochastic Processes, 2nd Edition" *Academic Press*, New York.

Karoonsoontawong, A. and S. T. Waller, 2006. "Dynamic Continuous Network Design Problem: Linear Bi-Level Programming and Metaheuristic approaches." *Transportation Research Record: Journal of the Transportation Research Board*, 1964, 104-117.

Kleinfeld, E. and Kleinfeld, M. (1997). "A Short Course in Matrix Theory." *Nova Science Publishers*.

Kriger, D., Shiu, S., and Naylor, S. (2006) Estimating toll road demand and revenue: A synthesis of highway practice. National Cooperative Highway Research Council, Synthesis No. 365, Washington, DC: Transportation Research Board, National Research Council.

Kunimoto, N., and Ikeda, M. (1992). "Pricing Options with Curved Boundaries." *Mathematical Finance,* 2(4), 275-298.

Lam, W.H.K, and Tam, M.L. (1998). "Risk analysis of traffic and revenue forecasts for road investment projects." *Journal of Infrastructure Systems,* 4(1), 19-27.

Lamond, B. and Stewart, N.F. (1981). "Bregman's balancing method". *Transportation Research Part B,* Vol. 15, No. 4, pp. 239-248.

Larsson, T., Lundgren, J., and Rydergren, C. (2001). "Most Likely Traffic Equilibrium Route Flows Analysis and Computation". *In Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*, 129-159, Kluver Academic Publishers, The Netherlands.

Law, A. (2007). "Simulation Modeling & Analysis." *McGraw Hill*, 4th Edition.

Lévy-Lambert, H. (1968). "Tarification des services á qualité variable: aplication aux péages de circulation." *Econometrica*, 36(3-4), 564-574.

Linsmeier, T.J. and Pearson, N.D. (2000). "Value at risk." *Financial Analyst Journal*, 56(2), 47-67.

Lo, H.K. and Y.K. Tung. (2003). "Network with Degradable Links: Capacity Analysis and Design." *Transportation Research Part B*, 37, 345-363.

Lu, C-C., Zhou, X., and Mahmassani, H. (2006). "Variable Toll Pricing and Heterogeneous Users: Model and Solution Algorithm for Bicriterion Dynamic Traffic Assignment Problem." *Transportation Research Record*, 1964, 19-26.

Lu, C-C., Mahmassani, H., and Zhou, X. (2007). "A bi-criterion dynamic user equilibrium traffic assignment model and solution algorithm for evaluating dynamic road pricing strategies." *Transportation Research Part C*, doi:10.1016/j.trc.2007.08.002.

Maher, M.J. and P.C. Hughes (1997). "A Probit-based Stochastic User Equilibrium Assignment Model." *Transportation Research Part B*, 31(4), 341-355.

Mahmassani, H., Zhou, X., and Lu, C-C. (2005). "Toll Pricing and Heterogeous Users: Approximation Algorithms for Finding Bicriterion Time-Dependent Efficient Paths in Large-Scale Traffic Networks." *Transportation Research Record*, 1923, 28-36 .

Marchand, M. (1968). "A note on optimal tolls in an imperfect environment." *Econometrica*, 36(3-4), 575-581.

May, A. and Milne D. (2000). "Effects of alternative road pricing systems on network performance." *Transportation Research Part A*, 34(6), 407-436.

Merton, R.C. (1973). "Theory of rational option pricing." *The Bell Journal of Economics and Management Science,* 4(1), 141-183.

Merton, R.C. (1974). "On the pricing of corporate debt: The risk structure of interest rate." *Journal of Finance,* 29(2), 449-470.

Mills, G. (1995). "Welfare and Profit Divergence for a Tolled link in a Road Network." *Journal of Transport Economics and Policy*, 29, 137-146.

Nagae,T., and Akamatsu, T. (2006). "Dynamic revenue management of a toll road project under transportation demand uncertainty." *Networks and Spatial Economics*, 6, 345-357.

Nevitt, P.K., and Fabbozzi, F. (1995). "Project financing*." Euromoney Publications PLC*, 6th Edition*,* London, UK.

Ortiz, I. N. and Buxbaum, J. N. (2008). "Protecting the Public Interest in Long-Term Concession Agreements for Transportation Infrastructure." *Public Works Management & Policy,* 13(2), 126-137.

Recker, W., Chung, Y, Park, J., Wang, L., Chen, A., Ji, Z., Liu, H., Horrocks, M., and Oh, J-S. (2005). Considering risk-taking behavior in travel time reliability. Institute of Transportation Studies, University of California, Irvine.

Ross, S. (1997). "Simulation." *Academic Press*, 2nd Edition, San Diego, California, USA.

Rossi, T, McNeil, S., and Hendrickson, C. (1989). "Entropy Model for Consistent Impact -Fee Assessment." *Journal of Urban Planning and Development*, 115(2), 51-63.

Rubinstein, M. and Reiner, E. (1991). "Breaking down the barriers." *Risk,* 4(8), 28-35.

Rich, D.R. (1994). "The Mathematical Foundations of Barrier Option-Pricing Theory." *Advances in Futures and Options Research,* 7, 267-310.

Santos, G. (2004). "Urban Congestion Charging: A Second Best Alternative". Journal of Transport Economics and Policy, Vol. 38, No.3, pp. 345-369.

Santos, G., Newbery, D. and Rojey, L. (2000). "Static vs Demand Sensitive Models and the Estimation of Efficient Cordon Tolls: An exercise for Eight English Towns." Paper No.01-0246, Department of Applied Economics, University of Cambridge.

Shao, H., Lam, W.H.K., and Tam, M.L. (2006). "A reliability-based stochastic traffic assignment model for network with multiple user classes under uncertainty in demand." *Networks and Spatial Economics*, 6, 173-204.

Sheffi, Y. and W. Powell. (1982). "An Algorithm for the Equilibrium Assignment Problem with Random Link Times." *Networks*, 12, 191-207.

Sheffi, Y. (1984). "Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods." *Prentice-Hall, Inc*., Englewood Cliffs, New Jersey.

Shepherd, S. and Sumalee, A. (2004). "A genetic algorithm based approach to optimal toll level and location problems." *Networks and Spatial Economics*, 4, 161-179.

Smith, M.J. (1979). "The Marginal Cost Taxation of a Transportation Network." *Transportation Research*, 13B, 237-242.

Sorge, M., and Gadanecz, B. (2004). The term structure of credit spreads in project finance, Working Papers No. 159, Bank for International Settlements, Basel, Switzerland.

Subprasom, K., Chen, A., and Kongsomsaksakul, S. (2003) "Optimal Selection of Capacity and Toll in a Build-Operate Scheme Under Multiple Uncertainties". *Journal of the Eastern Asia Society for Transportation Studies*, 5, 1155-1170.

Subprasom, K., and Chen, A. (2007). "Effects of regulation on highway pricing and capacity choice of build-operate-transfer scheme." *ASCE Journal of Construction Engineering and Management*, 133 (1), 64–71.

Sumalee, A. (2004). "Optimal Road User Charging Cordon Design: A Heuristic Optimization Approach." *Computer-Aided Civil and Infrastructure Engineering*, 19, 377-392.

Sumalee, A., May, T. and Shepherd, S. (2005). "Comparison of judgmental and optimal road pricing cordons." *Transport Policy*, 12, 384-390.

Sydsaeter, K., Strom, A., and Berck, P. (2005). Economists' Mathematical Manual. Fourth Edition, Springer, Berlin.

Transportation Research Board (2007). "Metropolitan Travel Forecast: Current Practice and Future Directions." *Transportation Research Board,* Special Report 288.

Trigeorgis, L. (1998). "Real Options: Managerial Flexibility and Strategy in Resource Allocation." *The MIT Press,* Cambridge, Massachusetts.

Tsai, J., and Chu, C. (2003). "The Analysis of Regulation on Highway Private Investment Under a Build-Operate-Transfer Scheme." *Transportation*, 30(2), 221-243.

Ukkusuri, S., Tom, V.M., and Waller, S.T. (2007). "Robust Network Design Problem under Demand Uncertainty." *Computer Aided Civil and Infrastructure Engineering,* 22, 6-18.

Vassallo, J, M. (2006). "Traffic Risk Mitigation in Highway Concession Projects: The Experience of Chile." *Journal of Transportation Economics and Policy,* 40(3), 359-381.

Verhoef, E., Nijkamp, P. and Rietveld, P. (1996). "Second Best congestion pricing: the case of an untolled alternative." *Journal of Urban Economics*, 40(3), 279-302.

Verhoef, E. (2002). "Second-Best congestion pricing in general static transportation networks with elastic demands." *Regional Science and Urban Economics*, 32(3), 281-310.

Viton, P. (1995). "Private Roads." *Journal of Urban Economics*, 37(3), 260-289.

Vickrey, W. (1969). "Congestion Theory and Transportation investment." *American Economic Review*, 59, 251-261.

Waller, S.T., Schofer, J.L., Ziliaskopoulos, A.K. (2001). "Evaluation with traffic assignment under demand uncertainty." *Transportation Research Record: Journal of the Transportation Research Board*, 1771, 69-75.

Waller, S.T. and Ziliaskopoulos, A.K. (2007). "A Chance-Constrained Based Stochastic Dynamic Traffic Assignment Model: Analysis, Formulation and Solution Algorithms." *Transportation Research Part C* (In Press).

Wardrop, J. (1952). "Some theoretical aspects of road traffic research." *Proceedings of the Institute of Civil Engineers*, 1(3), 325-378.

World Bank and Ministry of Construction, Japan (1999). "Asian Toll Road Development Program: Review of Recent Toll Road Experience in Selected Countries and Preliminary Tool Kit for Toll Road Development." *World Bank,* Washington D.C.

Yang, H., and Woo, K. (2000). "Competition and Equilibria of Private Toll Roads in a Traffic Network." *Transportation Research Record*, 1733, 15-22.

Yildirim, M.B., Hearn, D.W, (2005). "A first best toll pricing/next term framework for variable demand traffic assignment problems." *Transportation Research: Part B*, 39, 659-678.

Yang, H. and Meng, H. (2000). "Highway pricing and capacity choice in a road network under a build-operate-transfer scheme." *Transportation Research Part A*, 34, 207-222.

Yang, H., Tang, W.H., Cheung, W.M. and Meng, Q. (2002). "Profitability and welfare gain of private toll roads in a network with heterogeneous users." *Transportation Research Part A*, 36, 537-554.

Yescombe, E.R. (2002). "Principles of Project Finance." *Academic Press,* San Diego, California.

Zhang, X. and Yang H. (2004)."The optimal cordon-based network congestion pricing problem." *Transportation Research Part B*, 38, 517-537.

7. APPENDIX: CASE STUDIES

CASE STUDY 1: SIOUX FALLS NETWORK

The Sioux Falls network has been frequently used in the literature. However, it should be noted that the network representation that this research and the literature use is adjusted and simplified, thus the results presented here should not be regarded as an assessment of the effects of network improvement actions in the setting of the real Sioux Falls network.

Initial network

The focus of this case study is on the implementation of the developed methodology. The simplified version of the Sioux Falls network used here has 182 O-D pairs, with (original) link parameters and O-D trip table as described in Recker et al. (2005). Note that nodes 1, 2, 4, 5, 10, 11, 13, 14, 15, 19, 20, 21, 22, and 24 are assumed to be both origins and destinations (see Figure A-1). A situation is considered in which links 25 and 26 correspond to the toll road facility under consideration.

Figure A-1: Sioux Falls Network

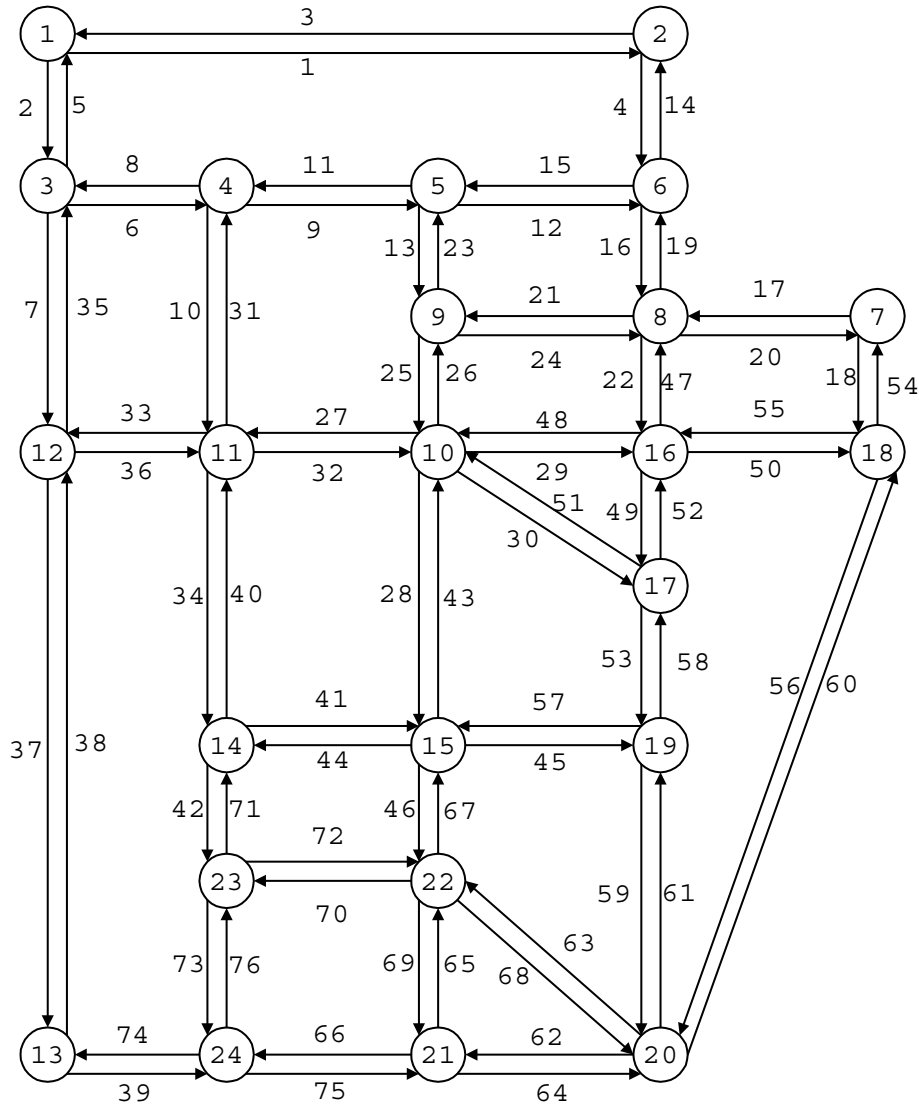## Step 1 – User Equilibrium

As explained in the body of the report, the first step in the developed valuation methodology is to determine the User Equilibrium (UE) link flows. Although the research team used its own code for this purpose, it should be noted that various software packages available in the market can be used here. Since the links 25 and 26 are chosen to represent the toll road, only

the traffic flows obtained on these two links are reported below as they are used as input values for the next step. The units are omitted for the purpose of generalizing the results.

Table A.1: User Equilibrium Solution for the Tolled Links

Tolled Link	Node		Link capacity	Traffic flow
	From	To		
25	9	10	13,920	9,140.01
26	10	9	13,920	9,139.85

It should be noted that the above traffic flows are considered to represent annual average daily traffic (AADT) on the tolled links. Since the developed methodology considers different financial instruments with regard to a time framework, that is, the risk assessment is based on future yearly predictions, yearly traffic (YT) is used in the calculation process. This traffic is determined as:

$$YT = AADT * 365 \qquad\qquad (A\text{-}1)$$

In summary, in the first step of the developed methodology, given the original network structure and the deterministic O-D demand, the UE traffic flows are determined. This solution gives information about the traffic flows on the tolled links at the initial time frame under consideration.

Step 2 – Forecast of Future Traffic/Revenue

The second step is to develop a stochastic model for forecasting the future traffic flow and revenue on the tolled links. This research uses geometric Brownian motion (GBM) for this purpose. The assumptions and mathematical formulation for this model are explained in detail in the body of the report. Thus, the focus on this section is on the implemented simulation method.

First, the initial value of the traffic flow and revenue on the tolled links must be obtained. From the previous step, the UE solution indicates that AADT on the tolled links 25 and 26 is **18,280** (9,140+9,140). In this case, YT is **6.67*E06** (18,280*365). In order to derive the initial

yearly revenue, it is assumed that the toll rate is constant over time and takes a unit value (i.e. $T_r = 1$). The yearly revenue is then R_0 =**6.67*E06** ($YT * T_r$).

The formula used for simulating GBM over time is as follows:

$$R_t = R_0 \exp^{\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]}$$

(A-2)

where R_t is the yearly revenue at some time $t \in [0, T]$, T being the time frame under consideration, μ is a drift rate, σ^2 is a variance, $dW_t = \sqrt{dt}\varepsilon_t$ is a Weiner process where dt is a time increment, and $\varepsilon_t \sim N(0,1)$. Thus, the future revenue on the tolled link can be fully defined by knowing its starting value R_0, expected growth rate μ, and volatility σ.

As the starting value for the yearly revenue is already known, the two remaining parameter values employed here are assumed based on values from the literature. The drift rate, which represents the expected traffic growth rate, is assumed as 5 percent. The volatility, which represents the standard deviation of the growth rate, is taken as 20 percent.

Since the developed methodology addresses three financial instruments (bank loans, bonds, and options), the time frame for the model simulation for all these instruments is also set here. Sorge and Gadanecz (2004) report an average maturity of 12 years for project finance loans based on a sample of loans with public sector guarantees, and 7.5 years for a sample of loans with and without guarantees. Thus, in this case study, the assumed length for debt service T is chosen as 10 and this time frame is used for all three instruments.

In summary, this step serves as the point at which predictions about future traffic flows should be considered as well as possible variations in these predictions. Also, the relationships between revenue and traffic flow are set. Once the input values for the model are determined (i.e. initial yearly revenue, expected traffic/revenue growth rate, volatility), Monte Carlo simulation is used for the simulation of the possible realizations over the time period under consideration. The next step correlates the behavior of the traffic flow on the tolled links with changes in the network structure, that is, changes on link capacities in the surrounding network.

Step 3 – Calculation of Elasticities

As defined in the body of the report, in this research, the sensitivity of the flow or revenue on a tolled link to a change in the capacity on another network link is assessed through the estimation of elasticities. Denote the set of links that constitute the toll road facility as I and (as before) the set of all network links as A. Thus, equation (4-31b) can be expressed in terms of yearly revenue, so that the elasticity ε_{R_i,C_j} of the yearly revenue R_i on the tolled link i ($i \in I$) to a change in the capacity C_j on link j ($i \neq j$, $j \in A$, $j \notin I$) can be expressed as follows:

$$
\varepsilon_{R_i,C_j} = \frac{\dfrac{\Delta R_i}{R_i^0}}{\dfrac{\Delta C_j}{C_j^0}}
\tag{A-3}
$$

where ΔR_i is the change in yearly revenue on the tolled link i as a consequence of the change in capacity on link j, R_i^0 is the initial yearly revenue on the tolled link i, ΔC_j is the planned change in the capacity on link j, and C_j^0 is the initial value of the capacity on link j.

The information of interest in this step is the size of the change in the yearly revenue (i.e. increase or decrease in the yearly revenue) as a result of the change on the capacity of some link in the network. Denoting the yearly revenue on the tolled link i after the occurrence of a change in the network structure as R_i', its value can then be determined as:

$$
R_i' = R_i^0 + \Delta R_i = R_i^0\left(1 + \varepsilon_{R_i,C_j} * \frac{\Delta C_j}{C_j^0}\right)
\tag{A-4}
$$

Thus, given the previously obtained elasticity value, one can determine the magnitude of the change in the yearly revenue due to a planned network improvement action and the value for the corresponding yearly revenue after the change. Note that the size of the change in the initial capacity is treated here as either a percent increase or decrease in the initial capacity.

In this case study, the elasticities are calculated for marginal changes in the capacities on selected links. Based on the previously introduced methodology for identification of feeder and competing links, four links were selected as competing links to the tolled links 25 and 26, while four links were identified as feeders (See Figure A-2).
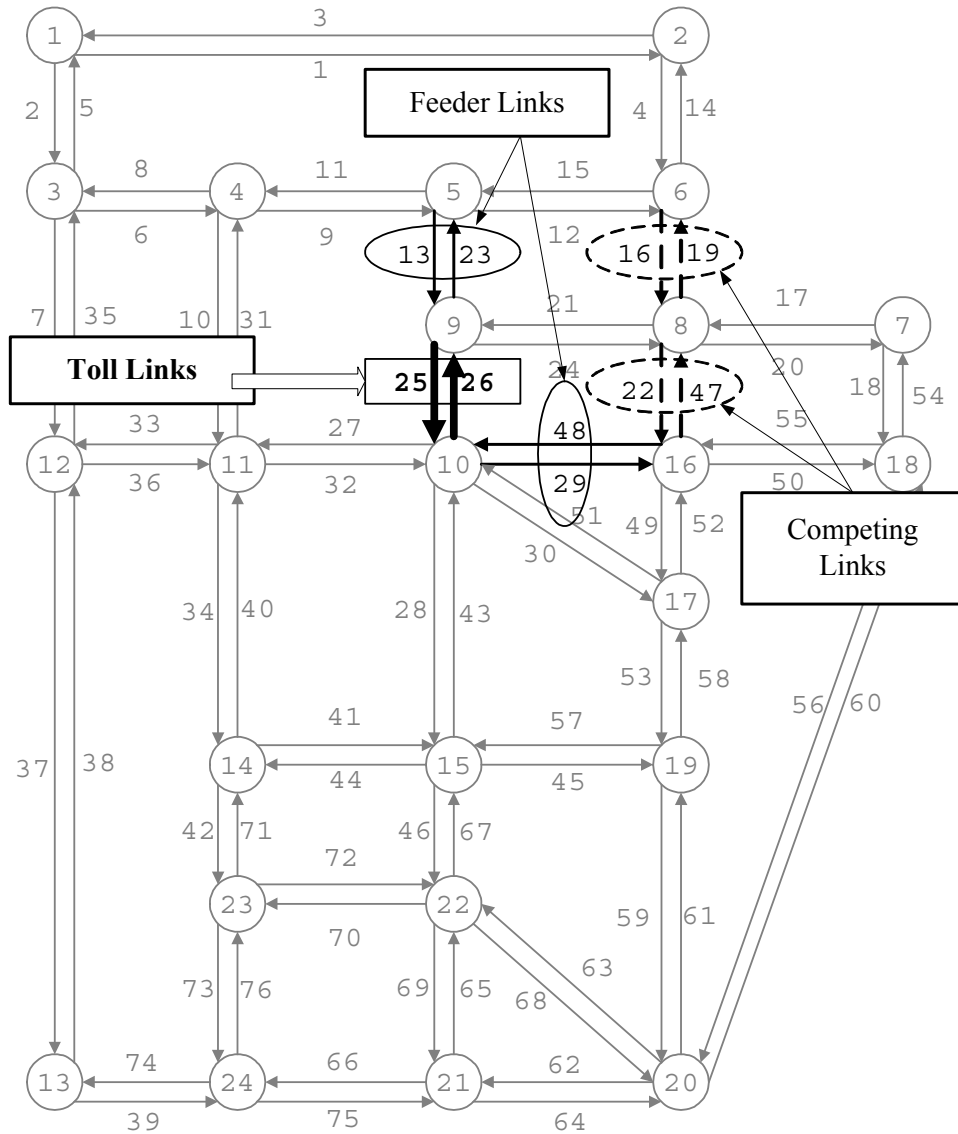
Figure A-2: Selected Links in Sioux Falls Network

Next, the initial capacity on the selected links is changed in the range of [0.6, 1.4] of the initial capacity by considering steps of size 0.1. This represents the effect of network improvement actions ranging between a 40 percent decrease in the initial capacity and a 40 percent increase in the initial capacity. The capacities on the selected links were changed one at a time, the corresponding UE flows were obtained, and the values for the (resulting) traffic flows on the tolled links 25 and 26 were recorded. Next, based on these new traffic flows, the new

yearly revenue on the tolled links is calculated. These values are then used to calculate the elasticities.

Before introducing the values for the calculated elasticities, one more calculation step needs to be mentioned. Since the link capacities on selected links were changed one at a time to calculate the elasticity on either the tolled link 25 or 26 to a change in the capacity on one of the selected feeder or competing links, one must also determine the corresponding elasticity of the joint yearly revenue (i.e. yearly revenue on both tolled links). To do so, Equation (4-31c) is used as follows:

$$\varepsilon_{R_{25}+R_{26},C_j} = \frac{R_{25}^0 * \varepsilon_{R_{25},C_j} + R_{26}^0 * \varepsilon_{R_{26},C_j}}{R_{25}^0 + R_{26}^0} \quad j = 13, 16, 19, 22, 23, 29, 47, 48 \tag{A-5}$$

The following table shows the calculated elasticities for the selected links.

Table A.2: Elasticities of the Joint Revenue on Tolled Links 25 and 26 to a Change in the Capacity on a Selected Link

	Change in the initial capacity								
	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
Feeder link 13	0.196127	0.120759	0.133524	0.119547	0.000000	0.044327	0.037093	0.031933	0.027496
Feeder link 23	0.196364	0.120584	0.133770	0.119793	0.000000	0.045547	0.036362	0.031847	0.027904
Feeder link 29	0.011328	0.015396	0.018023	0.014798	0.000000	0.009409	0.007768	0.007555	0.006392
Feeder link 48	0.011937	0.015573	0.017861	0.016346	0.000000	0.009759	0.008715	0.007699	0.006197
Competing link 16	-0.219690	-0.214444	-0.205037	-0.193858	0.000000	-0.046691	-0.023687	-0.015520	-0.022195
Competing link 19	-0.219572	-0.214070	-0.203834	-0.192693	0.000000	-0.045701	-0.023272	-0.015164	-0.022161
Competing link 22	-0.072529	-0.060405	-0.051614	-0.042243	0.000000	-0.028994	-0.023225	-0.015347	-0.011738
Competing link 47	-0.072611	-0.060145	-0.050854	-0.042681	0.000000	-0.028365	-0.021923	-0.015197	-0.011251

From Table A-2, it is observed that that the feeder links have associated positive elasticity values while the competing links have negative elasticity values associated with them. Note that, these results are not surprising given the method for identifying competing and feeder links proposed here and the logic behind the method. The next step in the developed methodology is to introduce the changed yearly revenue into the model and observe the behavior of the financial instruments with the respect to changes on the capacities of selected links.

Step 4 – Changes in the financial parameters

In this step, sensitivity analyses are conducted to assess the impact of changes in the network structure (i.e. changes on link capacities) on selected financial instruments. Recall that the toll road consists of links 25 and 26 and that the links selected for studying the impact of network improvement actions are the feeder links 13, 23, 29, 48 and the competing links 16, 19, 22, and 47.

Before discussing the results of the sensitivity analyses, let us summarize all the previous steps in this case study. For the given initial network structure and deterministic O-D trip matrix, the UE traffic flows on all the network links were obtained. Focusing only on the tolled links, the initial revenue was derived based on the obtained traffic flows on both tolled links and the given toll rate. Further, in order to address the revenue risk as one of main risk drivers in the financial analysis of toll roads, the stochastic model for forecasting revenue was introduced. The purpose of this model is to capture possible future outcomes of the revenue stream and define, in mathematical terms, the uncertainty in these future realizations.

Following main principles for pricing the revenue risk associated with selected financial instruments (i.e. bank loans, bonds, and options), the price of the revenue risk can be derived by considering both the previously introduced model for forecasting the revenue stream and the arranged debt service. In the case of bank loans and bonds, this price reflects the risk that the borrower will not be able to service the debt as planned. In the case of options, this price represents the price of the right or the obligation to exercise or accept the conditions from a contract between two parties. In this case, the type of option under consideration is the option to buy back the project, which grants the right to the public sector to buy back the toll road project if the revenue is above some agreed value.

Then, the relevant question here is how these prices of the revenue risk for these different financial instruments are affected by changes in the network structure. For instance, let us assume that the toll project starts its operation now and that a decision about the improvement on one of the network links is made two years from now. Since the financial agreements are already set, the decision about changing one link will affect the price of the risk for the debt

(which was set already). The methodology proposed in this research allows one to assess how the changes on the capacity of some network links affect this price.

The following subsections present the results of the sensitivity analyses for the Sioux Falls network. First, the results corresponding to the financial parameters used to price risk for bank loans are presented. Then, the results for bonds and options are discussed.

Bank Loans

The input values for Monte Carlo simulation are presented in Table A-3. Monte Carlo simulation is used for simulating revenue paths given the initial value R_0. The ratio of the starting revenue R_0 to the debt level D_l is set to 1.4, as this level is consistent with annual debt service coverage ratios for standard toll road projects (Yescombe, 2002). The number of simulated paths for the revenue process (i.e. number of possible revenue path realizations) is $n = 100,000$.

Table A.3: Summary of Input Parameters for Monte Carlo Simulation for Loans

T	R_0	μ	σ	T_r	$\dfrac{R_0}{D_l}$	n
10	$6.67 * 10^6$	0.05	0.2	1	1.4	100,000

It is expected that the increase in the capacity on a feeder link will positively affect the risk parameters, that is, add to the value of the project. As the capacity on a feeer link increases, the travel cost on the link decreases, thus attracting more traffic. As the number of trips on the feeder link increases, the traffic on the tolled link will also increase. Thus, the higher the revenue collected on the toll road, the lower the probability of defaulting on the debt service and the lower the risk weight on the asset. On the other hand, the increase in the capacity on a competing link is expected to detract from the value of the project.
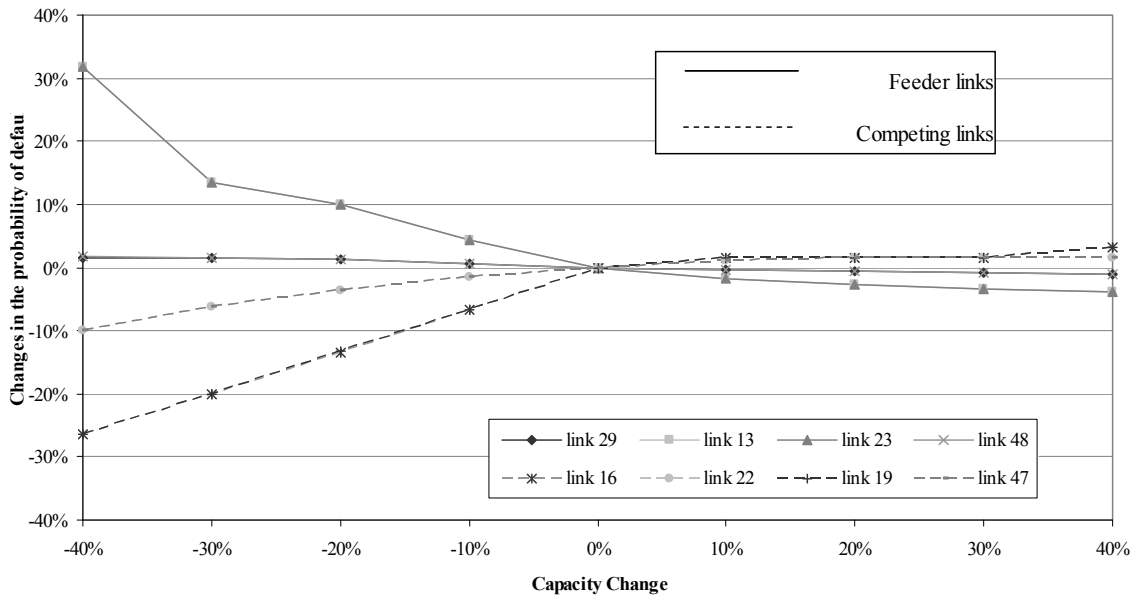
Figure A-3: Changes in the Probability of Default in a Five Year Period

Figure A-3 illustrates the dependency between the probability of default (PD) and the changes in capacities on feeder and competing links. The horizontal axis represents the percent change on the link capacity while the vertical axis represents the probability that the project will default within five years of debt service. The results show expected trends: an increase/decrease in the capacity on a feeder link decreases/increases the price of the revenue risk, thus reducing/increasing PD; while an increase/decrease in the capacity on a competing link increases/decreases the price of the risk, thus increasing/decreasing PD.

Surprisingly, the sensitivity of PD to changes that lead to a decrease in link capacity is higher than the sensitivity to changes that lead to an increase in link capacity. For instance, when the capacity on the feeder links 23 or 13 is decreased by 40 percent, PD will increase about 30 percent. However, if the capacity on either link 23 or 13 is increased by 40 percent, PD will decrease four percent only.

The magnitude of the impact of capacity changes on PD depends on the associated elasticity. In other words, if the elasticity of the flow/revenue on the tolled links to a change in the capacity on the selected link is high, the impact of the change in the link capacity on PD will also be high. This can be confirmed by observing the results for the elasticities presented in

Table A-2 and the results shown in Figure A-3. For example, the impact on PD due to a change in the capacity on the feeder link 13 is higher than the impact due to a change in the capacity on the feeder link 29, because the elasticity of the traffic volume/revenue on the tolled links to a change in the capacity on the link 13 is higher than the corresponding value associated with link 29.

Figure A-4 illustrates the effect of changes in the network structure on risk weights. The horizontal axis depicts the percent change on link capacity and the vertical axis measures the percent change on the risk weight for the asset (RWA). As expected, the results are in the line of the previously presented results for the probability of default. For instance, an increase in the capacity on a feeder link will lead to a decrease on the risk weight of the loan. On the other hand, an increase in the capacity on a competing link will result in an increase on the risk weight.

Similarly, as in the interpretation of results for the probability of default, it is observed that the incremental change on the risk weight of the asset due to a change in the capacity on a selected link is nonlinear and asymmetric. It is also noted that the impact on RWA depends on whether the selected link is a feeder or a competing link, and on the elasticity of the revenue/flow on the toll road to a change in the capacity on the selected link.
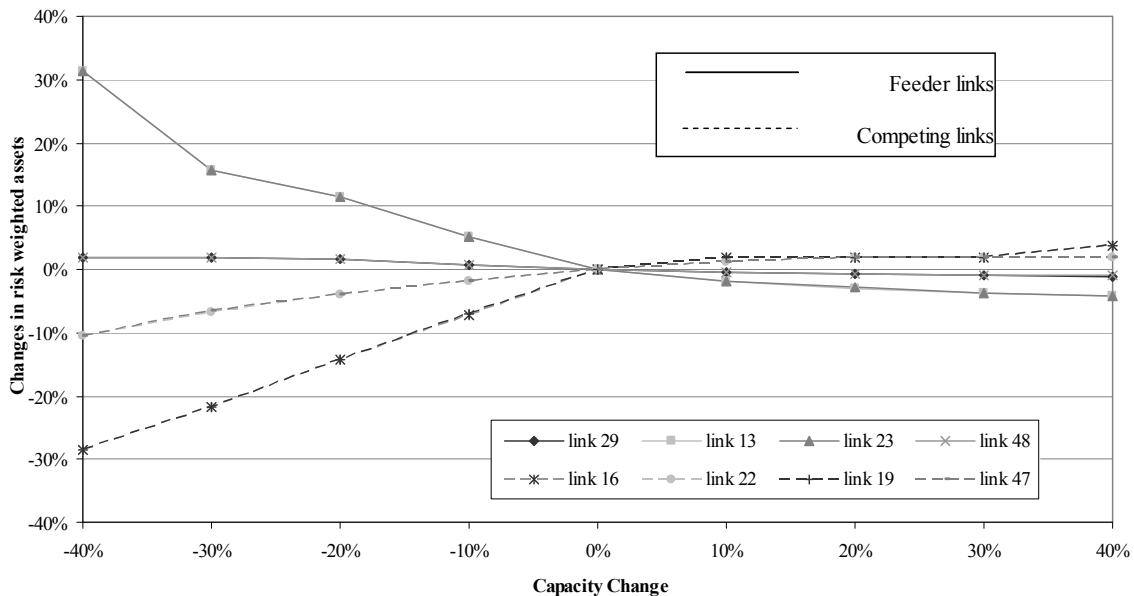

Figure A-4: Changes in Risk Weighted Assets

Bonds

As mentioned in the body of the report, the selected measure for the analysis of bonds is the credit spread. The higher/lower the credit spread, the higher/lower the risk associated with the underlying bond. In the context of this case study, it is expected that an increase in the capacity on a competing link will increase the credit spread, while an increase in the capacity on a feeder link will decrease it.

The input values for Monte Carlo simulation used in this sensitivity analysis are presented in Table A-4. As in the simulation for the sensitivity analysis for bank loans, the length of the debt service T_b is set to 10, so that the same base case is considered for all the financial instruments analyzed here. For the same reason, the other values are as previously used for bank loans. The additional input needed for assessing and testing the credit spread is the risk free rate r_f. Note that, although the identification of the risk free rate for toll roads is a topic of discussion (which is not addressed here), in the current application r_f is assumed to be equal to the drift rate value used for simulating GBM in a risk-free environment.

Table A.4: Summary of Input Parameters for Monte Carlo Simulation for Bonds

T_b	R_0	μ	σ	T_r	$\dfrac{R_0}{C_b}$	r_f	n
10	$6.67*10^6$	0.05	0.2	1	1.4	0.05	100,000

Figure A-5 presents the results of the sensitivity analysis for the credit spread. The horizontal axis shows the percent change in the capacity on a link and the vertical axis measures the percent change in the credit spread. The results are as expected: an increase in the capacity on a feeder link decreases the price of the revenue risk, thus adding to the value of the project; an increase in the capacity on a competing link increases the price of the risk, thus increasing the credit spread.
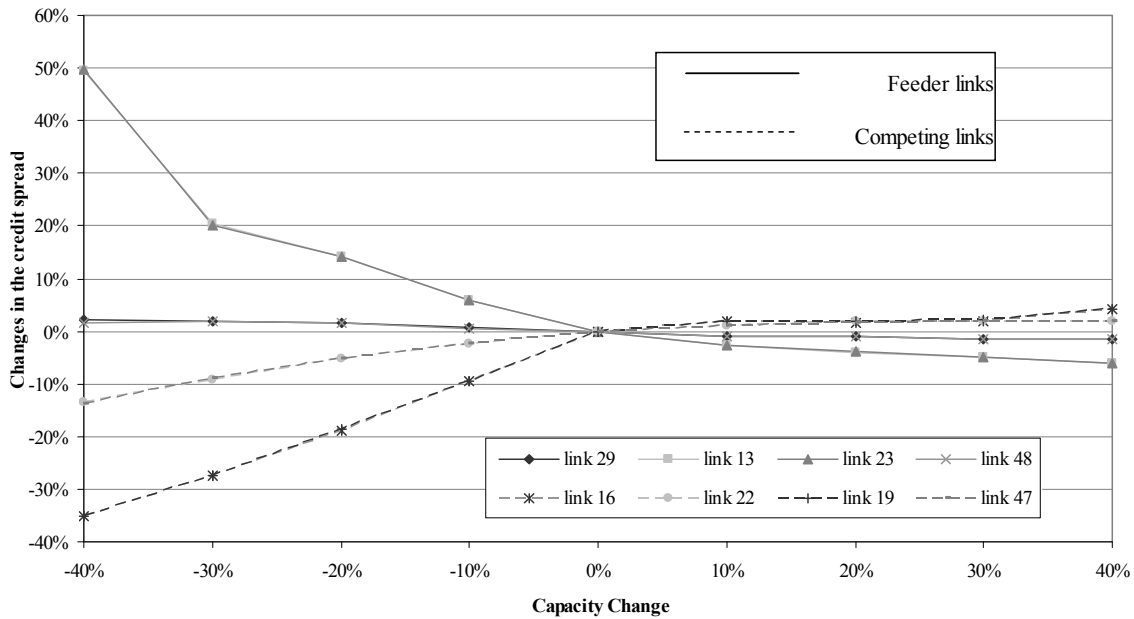
Figure A-5: Changes in the Credit Spread

As discussed when analyzing the results for bank loans, one must stress that the changes in the credit spread due to changes in the capacity on a feeder or competing link are asymmetric and non-linear. Also, the magnitude of the impact of a change in the capacity of a selected link on the credit spread depends on the role played by the link (i.e. whether the link is a competing or feeder link) and on the elasticity value associated with the selected link.

Buy-Back Option

The set of input values for Monte Carlo simulation is presented in Table A-5. These are the same values previously used for bank loans and bonds. The additional inputs needed for the assessment of the option's value is the upper bound UB, and the exercise price Kc. The number of simulations is $n = 100,000$.

Table A.5: Summary of Input Parameters for Monte Carlo Simulation for Buy Back Option

R_0	μ	σ	r_f	UB	Kc
$6.67*10^6$	0.05	0.2	0.05	$10*10^6$	$1.56*10^9$

It is expected that any increase in the capacity on a feeder link will increase the traffic flow and the revenue on the tolled links. Consequently, the expected value of the project will increase, resulting in an increased value of the option to buy-back the project. On the other hand, if the capacity on a feeder link decreases, the traffic flow and the revenue on the tolled links will decrease, thus decreasing the expected value of the project. Then, the price of the option to buy-back the project will decrease.
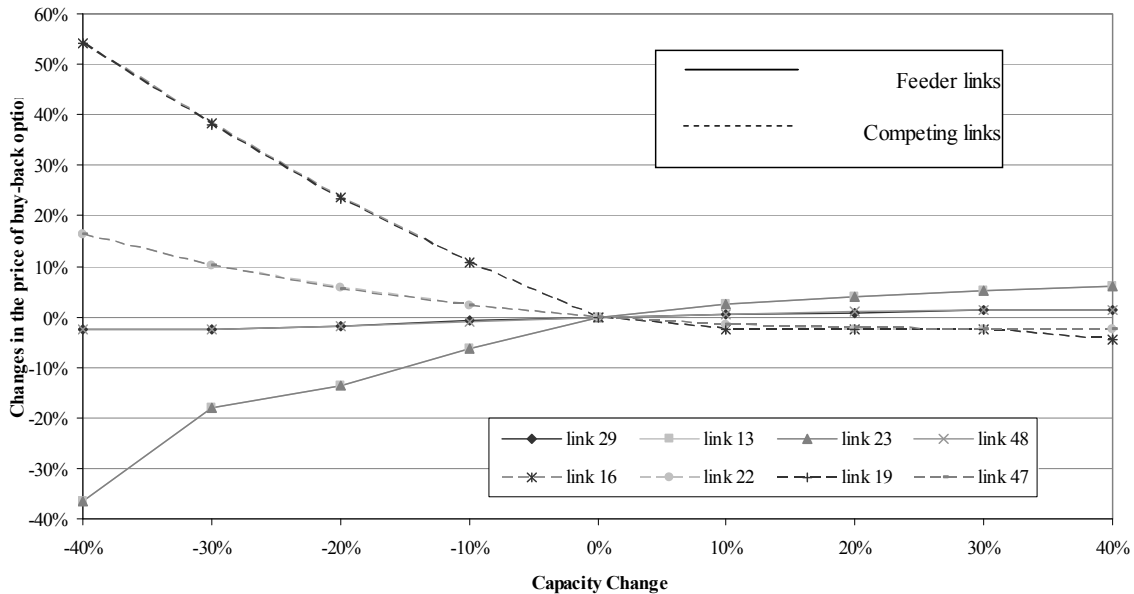
Figure A-6: Changes in the Price of the Buy Back Option

The results of the sensitivity analysis for the option to buy-back the project are presented in Figure A-6. The horizontal axis measures the percent change in the capacity on a selected link and the vertical axis shows the percent change in the option's value. As in the case of the previously studied financial instruments, an increase in the capacity on a selected link has a lower impact on the price of the buy-back option than a decrease in the link capacity. Again, the

magnitude of the impact depends on whether the change occurs on a competing or a feeder link and on the corresponding associated elasticity.

CASE STUDY 2: AUSTIN NETWORK

In this case study, the Loop 1N toll road is considered. The network improvement actions are assumed to occur on a non-tolled segment of Loop 1N. In this case, the planned network improvement actions correspond to a ten percent increase in the capacity on the links of the segment. Following the developed methodology, the results shown below are obtained for changes in the price of the revenue risk as a consequence of the planned network action.

Initial network

The toll road under consideration is the Loop 1N toll road which consists of 14 links with a total length of L=7.19 miles (blue ellipse in Figure A-7). The data for all the tolled links are given in Table A-6.

Table A.6: Loop 1N Toll Road

Link ID	Length (miles)	Toll rate (cents)	Capacity (veh/h)	Segment
7405	1.05	10.50	76,500	SH 45 - Parmer Ln (F)
7407	1.06	10.56	76,500	Parmer Ln - SH 45 (C)
7426	0.59	5.86	75,750	Parmer Ln - SH 45 (B)
7439	0.58	5.83	75,750	SH 45 - Parmer Ln (G)
7510	0.56	5.61	72,000	SH 45 - Parmer Ln (C)
7511	0.56	5.59	72,000	Parmer Ln - SH 45 (F)
7516	0.59	5.94	72,000	Parmer Ln - SH 45 (G)
7523	0.60	5.96	72,000	SH 45 - Parmer Ln (B)
7526	0.12	1.23	72,000	Parmer Ln - SH 45 (H)
7527	0.13	1.26	72,000	SH 45 - Parmer Ln (A)
7529	0.35	3.46	76,500	Parmer Ln - SH 45 (D)
7530	0.38	3.80	76,500	SH 45 - Parmer Ln (E)
7539	0.29	2.88	72,000	SH 45 - Parmer Ln (D)
7540	0.33	3.31	72,000	Parmer Ln - SH 45 (E)

Network improvement actions are considered for the non-tolled segment of Loop 1N between W Braker Ln and Duval Rd, which consists of 10 links with a total length of L=3.15 miles (shaded ellipse in Figure A-7). The initial capacities on these 10 links are given in Table A-7.

Table A.7: Non-Tolled Segment of Loop 1N

Link ID	Length (miles)	Capacity (veh/h)
34	0.43	76,500
5513	0.18	76,500
5508	0.56	76,500
5507	0.32	76,500
5387	0.43	76,500
5502	0.33	76,500
5492	0.21	76,500
5386	0.31	76,500
5488	0.19	76,500
5490	0.19	76,500

Figure A-7: Austin Network

Step 1 – User Equilibrium

First, the UE flows are obtained for the original network structure and O-D trip table. The corresponding UE link flows on the tolled links are shown in Table A-8.

Table A.8: UE Solution for Loop 1N Toll Road

Link ID	Flow (veh/day)
7405	63,298.42
7407	67,405.86
7426	88,371.23
7439	85,587.86
7510	49,057.23
7511	55,399.27
7516	59,119.25
7523	52,838.64
7526	59,119.25
7527	52,838.64
7529	77,973.61
7530	74,068.76
7539	74,068.76
7540	77,973.61
AADT	**937,120.39**

Based on the initial flow on the tolled links, the next step models and forecasts the future traffic flow/revenue on the toll road.

Step 2 – Forecast of Future Traffic/Revenue

As mentioned in the previous case study, the initial revenue R_0 is derived based on the determined average annual daily traffic (AADT). Since the toll rates are different for each segment of the Loop 1N toll road, the daily revenue is determined for each segment separately and then summed over all the tolled links. Thus, the total daily revenue for the Loop 1N toll road is **$48,062** which gives an initial yearly revenue (YT) of $R_0 = $17.5E06$.

The values for the drift rate (i.e. expected traffic growth rate) and the volatility of this rate are assumed. The traffic flow/revenue is expected to grow at a five percent rate per year and the variation of this value is assumed to be 20 percent. Based on the set of input values and Equation A-2, Monte Carlo simulation is used to simulate the revenue process as a process that evolves stochastically over time as GBM.

Step 3 – Calculation of Elasticities

As mentioned earlier, the ten links that were selected for the network improvement actions are part of a non-tolled segment of Loop 1N. It is assumed that there will be a 10 percent increase in the capacity on all of these non-tolled links. Thus, the elasticities of the flow/revenue on the tolled links to a change on the capacity on each of the ten non-tolled links were calculated separately (i.e. the capacity on each of these ten links was increased by 10 percent one at a time and the UE flows and corresponding revenue on the tolled links were recorded).

Since the planned network improvement action considers an increase in the capacities on all the links of the non-tolled segment, the elasticity of the (joint) toll road revenue to a capacity change on each of the ten links of the non-tolled segment are calculated by using Equation 4-31c for the case when the 14 tolled links of Loop 1 N are considered. Based on these ten elasticities, one can then calculate the corresponding change in the total toll road revenue due to individual marginal changes in the capacity on each of the ten links of the non-tolled segment. These changes in the total revenue are then summed over all the non-tolled links to predict/quantify the total impact on the yearly revenue when the capacities on all the links in the non-tolled segment are increased. Thus, the total impact of the 10 percent increase in the capacity of the non-tolled segment on the yearly revenue of the Loop 1N toll road is found to be $\Delta R = $**\$46,515.** The following step takes into account this revenue change and examines the impact of the assumed change in the network structure on selected financial measures.

Step 4 – Changes in the financial parameters

As in the previous case study, it is assumed that the planned increase in capacities occurs two years from now. The time span for the simulations is 10 years, and other assumed values

(risk free rate, debt coverage ratio) are as in the first case study. Since the sensitivity is determined only for a single planned action, the results for the three financial instruments are summarized in Table A-9.

Table A.9: Change in Financial Parameters after Network Improvement

ΔProbability of Default (1st year)	ΔRisk weighted assets (1st year)	ΔCredit spread	ΔPrice of the buy-back option
-0.6%	-0.5%	-1.0%	0.5%

As shown in the table, a 10 percent increase in the capacity on the above mentioned non-tolled segment of Loop 1N reduces the probability of default on the debt service, the weight of the risk in the bank's portfolio, and the spread on the price of the risky bond. On the other hand, the price of the option to buy-back the project is increased. These results are consistent with the results from the previous case study as the non-tolled segment of Loop 1N act as feeder to the tolled road, and the elasticities of the toll road revenue to a change in the capacity of the links of the non-tolled segment are positive.

However, one must stress that the impact on the financial instruments due to the change on the capacity of the links of the non-tolled segment is not significant. As mentioned earlier, the magnitude of the impact depends on the position of the links which are selected for changes in the network and the value of their associated elasticities. Thus, since the toll road revenue shows a low sensitivity to changes on the capacity of the links on the non-tolled segment, the magnitude of the impact is very modest. Of course, the impact will vary if a different network action is considered.

8. APPENDIX: IMPLEMENTATION GUIDE

This section describes the steps required for replicating the methodology proposed in this research. Note that, since the traffic assignment problem and identification of feeder and competing links have already been explained in detail in sections 4.1 and 4.3, this guide does not discuss such topics. Thus, it is assumed that the subset of links subject to capacity changes has already been selected. The focus here, therefore, is on explaining the simulations and computations required for quantifying the impact of changes in the network structure on the value of the toll road project under consideration.

Step 2: Forecast of Future Revenue on the Toll Road

Inputs:

o R^0: Initial revenue on the toll road. R^0 is calculated based on the traffic volumes on the tolled links obtained from Step 1 (traffic assignment)

o μ : Drift rate (expected growth rate of the revenue on the toll road)

o σ : Volatility (standard deviation of the expected growth rate)

o T : Life of the project

o np : Number of desired revenue paths to be simulated

o $step$: Time increment to be considered

o nt : Number of time intervals to be considered during the project's life (i.e. $nt=T/step$)

Output:

o $\mathbf{R}$: ($nt + 1$) x np matrix of revenue realizations on the toll road

This step aims at forecasting the uncertain future revenue on the toll road by assuming that the revenue follows Geometric Brownian motion (GBM). Below we show how this can be accomplished by simulating GBM over the desired time frame.

(i) Generate an nt x np matrix of standard normal random variables and call it **RND**

(ii) Based on the **RND** matrix, obtain an *nt* x *np* matrix, **A** , via the cumulative sum of each column of **RND** over its rows. That is, denoting the rows of **RND** as *i* and its columns as *j*, Matrix **A** can be obtained as follows:

$$\mathbf{A}(i, j) := \mathbf{RND}(i, j) \qquad\qquad i=1 \qquad\qquad j=1,\ldots,np$$

$$\mathbf{A}(i, j) := \mathbf{RND}(i, j) + \mathbf{RND}(i-1, j) \quad i=2,..,nt \quad j=1,\ldots,np$$

(iii) Compute an *nt* x *np* matrix, **I** , as $\mathbf{I} = \sigma * \sqrt{step} * \mathbf{A}$

(iv) Generate an *nt* x *np* matrix, **J** , by making the *np* columns of **J** equal to the *nt* x 1 vector $\mathbf{V} = \left(\mu - \dfrac{\sigma^2}{2}\right) * step * \mathbf{K}$, where **K** denotes an *nt* x 1 column vector which contains the end points of the time intervals in which the project's life has been divided (i.e. $\mathbf{K} = (1,2,\ldots\ldots, nt)'$).

(v) Compute an *nt* x *np* matrix, $\mathbf{R}_{en}$, of forecasted revenues at the end of the considered time intervals as:

$$\mathbf{R}_{en} = R^0 * \exp(\mathbf{I} + \mathbf{J})$$

(vi) Create an (*nt* + *1*) x *np* matrix, **R** , of forecasted revenue on the toll road (including the initial time period), as:

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_{R0} \\ \mathbf{R}_{en} \end{pmatrix}$$

where $\mathbf{R}_{R0}$ is a *1* x *np* row vector with its *np* positions equal to R^0 and $\mathbf{R}_{en}$ as defined above. Thus, denoting the rows of the matrix **R** as *i* and the columns as *j*, the $\mathbf{R}(i, j)$ element of **R** corresponds to the revenue realization at time *i* (*i*=0,1,….,*nt*) under the *j*-th sampled revenue path (*j*=1,….,*np*). Note that the **R** matrix is an input for the calculation of the selected financial risk parameters described in Step 4 of this implementation guide.

Step 3: Impact of Improvement Actions on Competing and Feeder Links

Inputs:

o R_i^0: Initial revenue on the tolled link $i \in I$, where I is the subset of links that constitute the toll road. R_i^0 is calculated based on the (initial) traffic flow on link i, v_i^0, obtained from traffic assignment under the original network structure

o C_j^0: Initial capacity on link j ($j \in J$), where J is the subset of links on which an improvement action is planned

o C_j': Final capacity on link j ($j \in J$) after the proposed change in its capacity

Output:

o   Quantification of the change in total revenue on the toll road as a result of improvement actions on a subset of J network links. In particular, a parameter z is obtained which, given the occurrence of the above mentioned network change, is later used to update the random revenue path realizations that were previously obtained in Step 2.

Here it is assumed that the subset of J links subject to capacity changes has already been selected by using the methodology for identifying competing and feeder links as described in Section 4.3 of the report. Note, however, that the methodology proposed in this research can be used to study the effect of changes in the capacity on any subset of J network links, regardless of how the subset is determined. Below we calculate relevant information for updating the previously generated toll road revenue realizations after a change in the network structure has occurred.

(i) Calculate the elasticity of the revenue on the tolled link i ($i \in I$) to a change in the capacity on link j ($j \in J$):

Denote the set of links that constitute the toll road as I, the set of all network links as A, and the set of links on which capacity changes are planned as J. The elasticity ε_{R_i, C_j} of the revenue R_i on the tolled link i ($i \in I$) to a change in the capacity on link j ($i \neq j, j \in A, j \notin I, j \in J$) can be obtained as:

$$\varepsilon_{R_i,C_j} = \frac{\dfrac{\Delta R_i}{R_i^0}}{\dfrac{\Delta C_j}{C_j^0}}$$

where ΔR_i is the change in revenue on the tolled link i due to a change in the capacity on link j, and ΔC_j is the proposed change in the capacity on link j (i.e. $\Delta C_j = C_j' - C_j^0$). Note that ΔR_i ($\Delta R_i = R_i' - R_i^0$) is obtained based on the link volumes v_i^0 and v_i', resulting from two traffic assignments: the first under the original capacity on link j, C_j^0, and the second under the modified capacity, C_j'.

(ii) Calculate the elasticity of the total toll road revenue to a change in the capacity on link j ($j \in J$):

Note that, if the toll road facility consists of S links (i.e. $|I| = S$), then the elasticity ε_{R^0,C_j} of the total revenue R^0 on the toll road ($R^0 = \sum_{i=1}^{S} R_i^0$) to a change in the capacity on link j ($j \in J$) can be obtained as:

$$\varepsilon_{R^0,C_j} = \frac{\sum_{i=1}^{S} R_i^0 * \varepsilon_{R_i,C_j}}{\sum_{i=1}^{S} R_i^0}$$

where R_i^0 is the initial revenue on link i ($i=1,\ldots,S$) and ε_{R_i,C_j} ($i=1,\ldots,S$; $j \in J$) is calculated as described in step (i) above.

(iii) Quantification of the total toll road revenue after a change in the capacity on a subset of J network links has occurred:

Note that, given the initial toll road revenue, R^0, the new toll road revenue, R' ($R' = \sum_{i=1}^{S} R_i'$), resulting from a change in the capacity on a subset of J network links can be expressed as:

$$R' = R^0 + \Delta R$$

where the aggregate change in toll road revenue, ΔR, can be written as:

$$\Delta R = \sum_{j \in J} \Delta R_j$$

where ΔR_j is the change in total toll road revenue due to a change in the capacity on link j ($j \in J$). Now, noting that:

$$\Delta R_j = R^0 * \varepsilon_{R^0, C_j} * (\Delta C_j / C_j^0)$$

the new total toll road revenue after the occurrence of a change in the network structure can then be estimated as:

$$R' = R^0 * [1 + \sum_{j \in J} \varepsilon_{R^0, C_j} * (\Delta C_j / C_j^0)]$$

or $\qquad R' = R^0 * [1 + z]$ where $z = \sum_{j \in J} \varepsilon_{R^0, C_j} * (\Delta C_j / C_j^0)$

Note that the estimated z parameter is an input for Step 4. z is used to update the simulated revenue realizations obtained in Step 2, after a change in the network structure occurs.

Step 4: Assessment of the Effect of Network Changes on Risk Dependent Value Measures

Inputs:

- D: Scheduled debt repayments. Note that, if the revenue realizations are calculated on a yearly basis (i.e. if $step = 1$ year), then total project debt is divided into equal yearly payments so $D =$ total project debt$/nt$

- $\mathbf{R}$: ($nt + 1$) x np matrix of simulated revenue realizations on the toll road as obtained in Step 2.

- r_f: Risk free rate used to calculate the present value of the option

- UB: Upper bound on the revenue used when calculating the option to buy back the project

- K_c: Exercise price for the option to buy back the project

- z: Parameter estimated in Step 3. Recall that $z = \sum_{j \in J} \varepsilon_{R^0, C_j} * (\Delta C_j / C_j^0)$

Output:

- Probability of default (*PD*) and Risk Weighted Assets (*RWA*) as risk measures for bank loans

- Credit spread as the selected risk measure for bonds

- Price of the option to buy-back the project

This step correlates the uncertain revenue with selected risk measures corresponding to the three financial instruments most commonly used in toll road concessions: bank loans, bonds, and real options. Since the proposed methodology quantifies the effect of network improvement actions on selected risk dependent value measures, one must calculate these risk measures for: (1) the base case (i.e. initial network structure), and (2) the modified network structure. The former is discussed in Step 4a while the latter is presented in Step 4b. The difference between the values from Steps 4a and 4b represents the marginal change in the selected risk measures due to a network improvement action. These analyses allow one to quantify the effect of the planned network improvement actions on the price of the revenue risk.

Step 4a: Calculation of Risk Measures for the Base Case (Original Network Structure)

Bank Loans

(i) Generate an *nt* x *np* zero matrix and denote it as **TEMP**. **TEMP** is then modified based on the **R** matrix by checking the revenue realization values corresponding to the first period and after (i.e. second row and up on matrix **R**) and by copying only those values that are below the debt service level, *D*, as follows:

$$\mathbf{TEMP}(i, j) := \mathbf{R}(i+1, j) \quad \text{if} \quad \mathbf{R}(i+1, j) \leq D \quad i = 1,..., nt \quad j = 1,...., np$$

Note that, if $\mathbf{R}(i+1) > D$, $\mathbf{TEMP}(i, j)$ will still be zero.

(ii) Generate a zero row vector, **K**, of dimension 1 x *np*. **K** will be used to store the time when the project defaulted under the *j*-th (*j*=1,..,*np*) revenue path realization. Note that, the *j*-th position of the **K** vector will be different from zero only if the toll road revenue $\mathbf{R}(i, j)$ dropped below the debt service level *D* at some time *i* and stayed below the debt service level until the end of the project life *T*. Then, the *j*-th (*j*=1,..,*np*) position of the **K** vector can be obtained based on the **TEMP** matrix as:

$$\mathbf{K}(1, j) := i \quad \text{if} \quad \mathbf{TEMP}(h) \neq 0 \quad \text{for all } h \text{ such that} \quad i \leq h \leq nt$$

Note that, if the project did not default under the *j*-th revenue path realization, $\mathbf{K}(1, j)$ will still be zero.

(iii) Generate a zero row vector, **REM_DEBT**, of dimension 1 x *np* to store the values of the remaining debt for those revenue path realizations under which the project defaulted.

(iv) Generate a zero row vector, **REM_REVENUE**, of dimension 1 x *np* to store the values of the remaining revenue for those revenue path realizations under which the project defaulted

(v) Set the desired time period, *t*, for which the probability of default and risk weight assets must be estimated

(vi) Update the **REM_DEBT** and **REM_REVENUE** row vectors based on the information contained in the row vector **K**. Note that if $\mathbf{K}(1, j) = t$ (i.e. if default occurred at time t under the j-th revenue path realization), then the j-th ($j=1,\ldots,np$) position of the above mentioned vectors can be calculated as:

$$\mathbf{REM\_DEBT}(1, j) = D*(T-t) \quad \text{and}$$

$$\mathbf{REM\_REVENUE}(1, j) = \sum_{i=\mathbf{K}(1,j)}^{T} \mathbf{R}(i+1, j)$$

Further, note that the number, m, of simulated revenue paths for which default occurred at time t can be obtained by counting the number of simulated revenue paths j ($j=1,..,np$) with $\mathbf{K}(1, j) = t$.

(vii)   Generate a 1 x np row vector, **RL**, to store the real losses. Compute the real losses for the j-th ($j=1,\ldots,np$) simulated revenue path as:

$\mathbf{RL}(1,j) = \mathbf{REM\_DEBT}(1, j) - \mathbf{REM\_REVENUE}(1, j)$

(viii)   Based on the distribution of the **RL** values obtained for the np simulated revenue paths, compute: (1) expected losses, EL, as the mean of the distribution of real losses, and (2) value at risk, VaR, as the 99.9 percentile of the distribution. Then, calculate the unexpected losses, UL, as $UL = VaR - EL$.

(ix) Compute the probability that the project will default in the year of interest, t, as the ratio between m obtained in step (vi), and the number of simulated scenarios: $PD = m / np$

(x) Compute risk weighted assets as $RWA = 12.5 * PD * UL$

Bonds

(i) Repeat steps (i) to (iv) of the procedure for the calculation of bank parameters

(ii) Let t_{dj} represent the time when default occurred under the *j*-th (*j*=1,…,*np*) revenue path simulation. Note that if $\mathbf{K}(1, j) = t_{dj} \neq 0$, the remaining debt and revenue for the *j*-th revenue path realization can be obtained as:

$$\mathbf{REM\_DEBT}(1, j) = D * (T - t_{dj}) \quad \text{and}$$

$$\mathbf{REM\_REVENUE}(1, j) = \sum_{i=\mathbf{K}(1,j)}^{T} \mathbf{R}(i+1, j)$$

(iii) Generate a 1 x *np* row vector, **PUT_OPTION** , to store the values of the put options (i.e. values of the risk that the bond holder will default on debt service under the revenue path realizations). Fill the *j*-th (*j*=1,…,np) position of the **PUT_OPTION** row vector as:

$$\mathbf{PUT\_OPTION}(1, j) = [\mathbf{REM\_DEBT}(1, j) - \mathbf{REM\_REVENUE}(1, j)] / (1 + r_f)^{t_{dj}}$$

(iv) Compute the value of the put option as the average of the option values over the simulated revenue paths:

$$value\_put\_option = \frac{\sum_{j=1}^{np} \mathbf{PUT\_OPTION}(1, j)}{np}$$

(v) Compute the value of the risk free bond as:

$$bond\_risk\_free = \sum_{i=1}^{T} \frac{D}{(1 + r_f)^i}$$

(vi) Compute the value of the risky bond as:

$$bond\_risky = bond\_risk\_free - value\_put\_option$$

(vii)   Compute the yield or rate of return , r_r , on the risky bond from the following equation:

$$bond\_risky = \sum_{i=1}^{T} \frac{D}{\left(1+r_r\right)^i}$$

(viii)   Compute the credit spread as the difference between the yield on the risky bond, r_r, and the risk free rate of return, r_f:

$$credit\_spread = r_r - r_f$$

Option to buy-back the project

(i) Set the time t_{bb} $(1 \le t_{bb} \le nt)$ when the option can be exercised

(ii) Generate a zero row vector, $\overline{\mathbf{AR}}$, of dimension 1 x np to store average revenue values over the time framework $[1, t_{bb}]$. For each simulated revenue path j ($j=1,\ldots,np$) compute the average revenue over the time framework $[1, t_{bb}]$ as:

$$\overline{\mathbf{AR}}(1, j) = \frac{\sum_{i=1}^{t_{bb}} \mathbf{R}(i, j)}{t_{bb}}$$

(iii) Generate a zero row vector, $\mathbf{EPV}$, of dimension 1 x np to store the expected project values under the np revenue path realizations.

(iv) Compute parameters h and v as

$$h = \frac{\sigma^2}{4}(T - t_{bb}) \quad \text{and} \quad v = \frac{2\mu}{\sigma^2} - 1$$

(v) Update the expected project value vector $\mathbf{EPV}$. Note that, for a given simulated revenue path j ($j=1,\ldots,np$), if $\overline{\mathbf{AR}}(1, j) > UB$, then the corresponding expected project value can be obtained as:

$$\mathbf{EPV}(1, j) = \frac{4\mathbf{R}(t_{bb}, j)}{\sigma^2} \frac{[\exp(2h(v+1)) - 1]}{2(v+1)}$$

(vi) Generate a row vector, $\mathbf{OPTION\_BB}$, of dimension 1 x np to store option values under the np revenue path realizations.

(vii) Update the $\mathbf{OPTION\_BB}$ row vector. Note that, for a given simulated path j ($j=1,..,np$), if $\mathbf{EPV}(1, j) > K_c$, then the value of the corresponding buy back option can be calculated as:

$$\textbf{OPTION\_BB}(1, j) = \frac{EPV(1, j) - K_c}{(1 + r_f)^{t_{bb}}}$$

(viii)   Compute the value of the option to buy back the project as the average of option values over all the simulated revenue paths:

$$value\_option\_bb = \frac{\sum_{j=1}^{np} \textbf{OPTION\_BB}(1, j)}{np}$$

Step 4b: Calculation of Risk Measures for the Planned Network Improvements

The following procedure explains how to determine the values for the selected credit risk measures as a result of the implementation of a planned network improvement action. The procedure shows how to determine the changes in the revenue realizations and how to incorporate these changes into the modeling framework.

(i) Set the time t' ($1 \leq t' \leq nt$) when the planned capacity change occurs

(ii) Generate an $(nt + 1)$ x np matrix, $\mathbf{R}'$, to store the updated revenue realizations on the toll road as a consequence of a change in the capacity on a subset of network links

(iii) Based on the matrix of revenue realizations, $\mathbf{R}$, which was previously obtained in Step 2, calculate the change in the revenue realizations due to changes in the network structure. To do so, one must modify only the realizations corresponding to period t' and after. Hence, for the j-th ($j=1,\ldots,np$) simulated revenue path, the updated revenue realization at time i ($i= t'$, t'+step,......,nt) can be computed as:

$$\mathbf{R}'(i, j) = \mathbf{R}(i, j) * [1 + z]$$

where the z term was calculated in Step 3 (impact of improvement actions on feeder and competing links).

(iv) Repeat all the steps in the procedures for the calculation of risk measures for bank loans, bonds, and the option to buy back the project by considering the modified matrix of revenue realizations $\mathbf{R}'$ instead of $\mathbf{R}$.

(v) Compute the difference between *PD*, *RWA*, credit spreads, and values of the option obtained in Steps 4a and 4b and record the change in the values as the change in the price of the revenue risk for the selected financial instruments.